

**SOME ASPECTS OF ANALYTIC NUMBER THEORY:
PARITY, TRANSCENDENCE, AND MULTIPLICATIVE
FUNCTIONS**

by

Michael J. Coons

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APPROVAL

Name: Michael J. Coons
Degree: Doctor of Philosophy
Title of thesis: Some aspects of analytic number theory: parity, transcendence, and multiplicative functions

Examining Committee: Dr. Steven Ruuth
Chair

Dr. Peter B. Borwein
Senior Supervisor, Simon Fraser University

Dr. Stephen K. K. Choi
Co-Supervisor, Simon Fraser University

Dr. Nils Bruin
Supervisory Committee, Simon Fraser University

Dr. Karen Yeats
Internal Examiner, Simon Fraser University

Dr. Michael A. Bennett
External Examiner, University of British Columbia

Date Approved: April 1, 2009

Abstract

Questions on parities play a central role in analytic number theory. Properties of the partial sums of parities are intimate to both the prime number theorem and the Riemann hypothesis.

This thesis focuses on investigations of Liouville's parity function and related completely multiplicative parity functions. We give results about the partial sums of parities as well as transcendence of functions and numbers associated to parities. For example, we show that the generating function of Liouville's parity function is transcendental over the ring of rational functions with coefficients from a finite field. Within the course of investigation, relationships to finite automata are also discussed.

In memory of Michael and William

*“A boundary between arithmetic and analytic
areas of mathematics cannot be drawn.”*

— *Neuer Beweis der Gleichung $\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0$,*

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*“If you want to climb the Matterhorn
you might first wish to go to Zermatt
where those who have tried are buried.”*

— *A note to a student working on
the Riemann hypothesis,*

GYÖRGY PÓLYA

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Preface

As the title would hopefully lead people to believe, this thesis is dedicated to developing some of the aspects of analytic number theory dealing with parity, transcendence, and multiplicative functions.

By parity, we mean just that, even or odd. We focus on the parity of the number of prime divisors of an integer. This idea is embodied, or enfunctioned, in the Liouville λ -function, which given an integer n , is defined to be 1 if the number of prime divisors of n , counting multiplicity, is even, and -1 if odd. By design, λ is completely multiplicative; that is, for all $m, n \in \mathbb{N}$ we have $\lambda(mn) = \lambda(m)\lambda(n)$.

Liouville's function is related to some very interesting theorems from prime number theory. Furthermore, the prime number theorem is equivalent to the statement that $\sum_{n \leq x} \lambda(n) = o(x)$, and the Riemann hypothesis is equivalent to the statement that for every $\varepsilon > 0$, we have $\sum_{n \leq x} \lambda(n) = O(x^{1/2+\varepsilon})$.

Since an improvement on the asymptotic behaviour of $\sum_{n \leq x} \lambda(n)$ is beyond our grasp, we dwell upon some questions that we can answer, like “what about the partial sums of functions that are similar to Liouville's function?,” where “similar” will be determined later. We also address questions concerning the algebraic character of power series $\sum_{n \geq 1} f(n)z^n$ and special values of these series, where f is one of these “similar” functions.

To this end, this thesis is organized as follows.

Chapter 1 contains an introduction to the theory surrounding Liouville's function by providing a link to the classical theory of the distribution of primes. Included in this chapter is a new proof of a theorem of Landau and von Mangoldt, which states that *the prime number theorem is equivalent to* $\sum_{n \geq 1} \frac{\lambda(n)}{n} = 0$. We also give a new proof of the statement $\sum_{n \leq x} \lambda(n) = o(x)$ by providing a connection between the asymptotic density of a sequence and the residue of the zeta function associated to this sequence.

In Chapter 2 we broaden our focus by considering generalized versions of λ . In particular, define *the Liouville function for A* , a subset of the primes P , by $\lambda_A(n) = (-1)^{\Omega_A(n)}$ where $\Omega_A(n)$ is the number of prime factors of n coming from A counting multiplicity. For the traditional Liouville function, A is the set of all primes. Denote

$$L_A(n) := \sum_{k \leq n} \lambda_A(k) \quad \text{and} \quad R_A := \lim_{n \rightarrow \infty} \frac{L_A(n)}{n}.$$

Granville and Soundararajan [51] have shown that for every $\alpha \in [0, 1]$ there is an $A \subset P$ such that $R_A = \alpha$. Given certain restrictions on A , asymptotic estimates for $L_A(n)$ are also given. For *character-like functions* λ_p (λ_p agrees with a Dirichlet character χ when $\chi(n) \neq 0$) exact values and asymptotics are given; in particular

$$\sum_{k \leq n} \lambda_p(k) \ll \log n.$$

Within the course of discussion, the ratio $\varphi(n)/\sigma(n)$ is considered.

Chapter 3 contains an excursion into Mahler's method of proving transcendence which will be used heavily in Chapter 4. This method is used to prove the transcendence of power series which satisfy certain functional equations. This chapter is divided into two sections which deal with two canonically different types of functional equations. In the first section of this chapter, we give various transcendence results regarding functions related to the Stern sequence. In particular, we prove that the generating function of the Stern sequence is transcendental. Transcendence results are also proven for the generating function of the Stern polynomials and for power series whose coefficients arise from some special subsequences of the Stern sequence. In the second section, we prove that a non-zero power series $F(z) \in \mathbb{C}[[z]]$ satisfying

$$F(z^d) = F(z) + \frac{A(z)}{B(z)},$$

where $d \geq 2$, $A(z), B(z) \in \mathbb{C}[z]$ with $A(z) \neq 0$ and $\deg A(z), \deg B(z) < d$ is transcendental over $\mathbb{C}(z)$. Using this result and Mahler's Theorem, we extend results of Golomb and Schwarz on transcendental values of certain power series. In particular, we prove that for all $k \geq 2$ the series $G_k(z) := \sum_{n \geq 0} z^{k^n} (1 - z^{k^n})^{-1}$ is transcendental for all algebraic numbers z with $|z| < 1$. We give a similar result for $F_k(z) := \sum_{n \geq 0} z^{k^n} (1 + z^{k^n})^{-1}$.

In Chapter 4 we give a new proof of Fatou's theorem: *if an algebraic function has a power series expansion with bounded integer coefficients, then it must be a rational function*. This

result is used to show that for any non-trivial completely multiplicative function $f : \mathbb{N} \rightarrow \{-1, 1\}$, the series $\sum_{n \geq 1} f(n)z^n$ is transcendental over $\mathbb{Z}(z)$. For example, $\sum_{n \geq 1} \lambda(n)z^n$ is transcendental over $\mathbb{Z}(z)$, where λ is Liouville's function. The transcendence of $\sum_{n \geq 1} \mu(n)z^n$ is also proved. We continue by considering values of similar series. *The Liouville number*, denoted l , is the binary number

$$l := 0.100101011101101111100\dots,$$

where the n th bit is given by $\frac{1}{2}(1 + \lambda(n))$; here, as before, λ is Liouville's function. Presumably the Liouville number is transcendental, though at present, we know of no methods to approach proof. Similarly, define *the Gaussian Liouville number* by

$$\gamma := 0.110110011100100111011\dots$$

where the n th bit reflects the parity of the number of rational Gaussian primes dividing n , 1 for even and 0 for odd. In the second part of this chapter, using the methods developed in Chapter 3, we prove that the Gaussian Liouville number and its relatives are transcendental. One such relative is the number

$$\sum_{k \geq 0} \frac{2^{3^k}}{2^{3^{k+2}} + 2^{3^k} + 1} = 0.101100101101100100101\dots,$$

where the n th bit is determined by the parity of the number of prime divisors that are equivalent to 2 modulo 3.

In Chapter 5, using a theorem of Allouche, Mendès France, and Peyrière and many classical results from the theory of the distribution of prime numbers, we prove that $\lambda(n)$ is not k -automatic for any $k > 2$. This yields that $\sum_{n \geq 1} \lambda(n)X^n \in \mathbb{F}_p[[X]]$ is transcendental over $\mathbb{F}_p(X)$ for any prime $p > 2$. Similar results are proven (or reproven) for many common number-theoretic functions, including φ , μ , Ω , ω , ρ , and others.

Throughout Chapters 4 and 5, relationships to finite automata are discussed.

The sixth and final chapter of this thesis contains a collection of questions and conjectures for further study.

All of the results of this thesis have been published or submitted for publication. We have taken without hesitation from articles to which the author has been a major contributor ([13], [16], and [15]) or the sole author ([28], [29], [30], and [31]).

Chapter 1

Introduction

“Introduisons maintenant une fonction numérique nouvelle $\lambda(m)$, dont la valeur soit 1 ou -1 , suivant que le nombre total des facteurs premiers, égaux ou inégaux, de m est pair ou impair. En d’autres termes, soit $\lambda(1) = 1$, et généralement, pour m décomposé en facteurs premiers sous la forme $m = a^\alpha b^\beta \dots c^\gamma$, soit $\lambda(m) = (-1)^{\alpha+\beta+\dots+\gamma}$. Cette fonction $\lambda(m)$, prise isolément ou jointe à celles dont il a été question plus haut, donnera lieu à des théorèmes curieux.” [72]

1.1 Primes and parity

Recall that the *Liouville λ -function* is the unique completely multiplicative function for which $\lambda(p) = -1$ for all primes p . This function was already considered by Euler, 130 years before Liouville introduced the λ -notation.

In 1737, Euler stated the following theorems.

Theorem 1.1 (Euler [47]). *If we take to infinity the continuation of these fractions*

$$\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \dots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot 16 \cdot 18 \dots},$$

where the numerators are all the prime numbers and the denominators are the numerators less one unit, the result is the same as the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$$

which is certainly infinity.

Theorem 1.2 (Euler [47]). *If we assign a $-$ sign to all the prime numbers and composite numbers are assigned the sign that correspond to them according to the rule of signs in the product and with all the numbers we form the series*

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} - \frac{1}{12} - \dots$$

will have, once infinitely continued, sum 0.

In modern language, these theorems translate as follows.

Theorem 1.3. *In some infinite sense, one has that*

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-1} = \sum_{n \geq 1} \frac{1}{n},$$

and this series diverges.

Theorem 1.4. *Let $n = p_1^{k_1} \cdots p_r^{k_r}$ be the prime factorization of n ($n \geq 2$), $\Omega(n) = \sum_{j=1}^r k_j$, and $\lambda(n) = (-1)^{\Omega(n)}$ (using the convention that $\Omega(1) = 0$). In some infinite sense*

$$\sum_{n \geq 1} \frac{\lambda(n)}{n} = 0.$$

The words “in some infinite sense” are very important to the interpretations of these theorems. Indeed, as we will see later, one version of Theorem 1.4 is quite trivial and another is equivalent to the prime number theorem. We give modern proofs of both versions later in this chapter.

Theorem 1.1 introduces us to a very fundamental discovery in the theory of numbers: the zeta function with product formula. Although this was given by Euler (1737) many years before Riemann (1859), the zeta function is usually attributed to the latter, and the product formula to the former. In modern notation, we denote by $\zeta(s)$, the Riemann zeta function as a function of a complex variable, which for $\Re s > 1$ we have the representation,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad (1.1)$$

where the product is taken over all primes p .

Much is known about $\zeta(s)$. First we need to be able to view this function in a larger sense, in the whole complex plane. The standard way to analytically continue $\zeta(s)$ is to

begin with continuing $\zeta(s)$ to $\Re s > 0$ and then use a functional equation to complete the continuation to all of \mathbb{C} except for the point $s = 1$. For $\Re s > 1$ we have

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \sum_{n \geq 1} n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = s \sum_{n \geq 1} n \int_n^{n+1} x^{-s-1} dx.$$

Recall that $x = [x] + \{x\}$, where $[x]$ and $\{x\}$ are the integer and fractional parts of x , respectively. Since $[x]$ is always the constant n for any x in the interval $[n, n+1)$, we have

$$\zeta(s) = s \sum_{n \geq 1} \int_n^{n+1} [x] x^{-s-1} dx = s \int_1^{\infty} [x] x^{-s-1} dx.$$

Writing $[x] = x - \{x\}$ we have

$$\begin{aligned} \zeta(s) &= s \int_1^{\infty} x^{-s} dx - s \int_1^{\infty} \{x\} x^{s-1} dx \\ &= \frac{s}{s-1} - s \int_1^{\infty} \{x\} x^{-s-1} dx. \end{aligned}$$

We now observe that since $0 \leq \{x\} < 1$, the improper integral in the last equation converges when $\Re s > 0$ because the integral $\int_1^{\infty} x^{-\Re s-1} dx$ converges. Thus this integral defines an analytic function of s in the region $\Re s > 0$. Therefore the meromorphic function on the right-hand side of the last equation gives an analytic continuation of $\zeta(s)$ to the region $\Re s > 0$, and the $\frac{s}{s-1}$ term gives the simple pole of $\zeta(s)$ at $s = 1$ with residue 1.

We note the definition of the Γ -function.

Definition 1.5. For $\Re s > 0$,

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt. \quad (1.2)$$

For $s \in \mathbb{C} \setminus \mathbb{Z}$ the general Γ -function is given by

$$\Gamma(s) = \frac{-1}{2i \sin(\pi s)} \int_{\mathcal{C}} (-t)^{s-1} e^{-t} dt, \quad (1.3)$$

where the contour \mathcal{C} is oriented counter-clockwise and contains the nonnegative real axis.

The functions $\Gamma(s)$ and $\zeta(s)$ are related via a functional equation which completes the analytic continuation of $\zeta(s)$ to all of \mathbb{C} with the exception of $s = 1$.

Theorem 1.6 (Riemann [85]). *The function $\zeta(s)$ satisfies the functional equation*

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

As a function of a complex variable, $\zeta(s)$ is analytic everywhere except at $s = 1$. Now consider the zeros of $\zeta(s)$, and let us focus on the line $\Re s = 1$ away from the point $s = 1$; that is, the line $1 + it$ for $t \neq 0$. Since $\zeta(s)$ is analytic here, let us suppose that there is a zero on this line of order $r \geq 1$. Using Taylor series, we have that $\zeta(1 + \varepsilon + it) \approx c\varepsilon^r$ for $|\varepsilon|$ sufficiently small. Since $\zeta(s)$ has a pole of order 1 at $s = 1$, we know that $\zeta(1 + \varepsilon) \approx \frac{1}{\varepsilon}$.

We continue in the standard manner, using Mertens' simple identity

$$3 + 4 \cos(\theta) + \cos(2\theta) \geq 0.$$

Since

$$\Re \log \zeta(\sigma + it) = \sum_p \sum_{n \geq 1} \frac{\cos(t \log p^n)}{n \cdot p^{n\sigma}},$$

replacing t by $0, t, 2t$ in the above, one has that

$$3 \log \zeta(\sigma) + 4 \Re \log \zeta(\sigma + it) + \Re \log \zeta(\sigma + 2it) \geq 0,$$

so that for all real $\sigma > 1$ and $t \neq 0$,

$$\zeta^3(\sigma) |\zeta^4(\sigma + it) \zeta(\sigma + 2it)| \geq 1.$$

Substituting our ε -estimates in this equation we have

$$|c^4 \varepsilon^{4r-1} \zeta(1 + \varepsilon + 2it)| \geq 1.$$

Taking the limit as $\varepsilon \rightarrow 0$ implies that $\zeta(s)$ has a pole of order $4r - 1 \geq 1$ at $s = 1 + 2it$, contradicting the fact that $\zeta(s)$ is analytic there. Hence we have shown

Theorem 1.7 (Hadamard and de la Vallée Poussin, 1896). $\zeta(s) \neq 0$ on the line $\Re s = 1$.

1.2 The prime number theorem

The prime number theorem states that

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x / \log x} = 1 \tag{1.4}$$

where $\pi(x)$ denotes the number of primes less than or equal to x . This was Gauss' original formulation, which was proved independently by Hadamard [53] and de la Vallée Poussin [32] in 1896. They proved this by showing that $\zeta(s) \neq 0$ in the region $\Re s \geq 1$, where $\zeta(s)$ is the Riemann zeta function. Indeed, one has that

Theorem 1.8 (Hadamard [53], de la Vallée Poussin [32]). *The prime number theorem is equivalent to the non-vanishing of $\zeta(s)$ in the region $\Re s \geq 1$.*

One may also read the prime number theorem, as given by Landau, in the following way: *asymptotically there is an equal probability that a given number is the product of an even or an odd number of primes, with multiple factors counted with multiplicity* [67, p. 630].

To formalize this statement, consider the following theorem of von Mangoldt. But first, recall that the *Möbius function* $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$ is given by

$$\mu(n) := \begin{cases} 1 & n = 1, \\ 0 & \text{if } k^2 \mid n \text{ for some } k \geq 2, \\ (-1)^r & n = p_1 p_2 \cdots p_r. \end{cases}$$

Theorem 1.9 (von Mangoldt [94]). *The prime number theorem implies that*

$$\sum_{n \geq 1} \frac{\mu(n)}{n} = 0.$$

Landau [65] gave a new proof of von Mangoldt's result, again using the prime number theorem, and also proved the converse of Theorem 1.9 [66]. Included in these works, he showed that

Theorem 1.10 (Landau [65, 66]). *The prime number theorem gives*

$$\sum_{n \leq x} \mu(n) = o(x). \tag{1.5}$$

In his “Handbuch” [67], Landau gave proofs of these theorems with the Liouville function in place of the Möbius function.

The traditional way to prove the prime number theorem is via Theorem 1.8. The statements in Theorems 1.9 and 1.10 are much less widely known, though they are of a more intuitive nature. For the remainder of this introduction, we provide new proofs of the λ -analogues of Theorems 1.8 and 1.9. More formally, we prove the following theorems.

Theorem 1.11. *The following are equivalent:*

- (i) $\zeta(s) \neq 0$ when $\Re s \geq 1$,
- (ii) $\sum_{n \geq 1} \frac{\lambda(n)}{n} = 0$.

Theorem 1.12. *Let λ denote Liouville's function. Then $\sum_{n \leq x} \lambda(n) = o(x)$.*

To emphasize Landau's quote (see page v of this thesis), the theorems, proofs, and methods contained in this chapter are intended to highlight the rich interplay between the arithmetic and analytic areas of mathematics.

1.2.1 A useful equivalence

Let A be a subset of \mathbb{N} and denote by $A(n)$ the number of elements in A that are less than or equal to n . When it exists, the *asymptotic density* of A in \mathbb{N} , denoted $d(A)$, is given by

$$d(A) := \lim_{n \rightarrow \infty} \frac{A(n)}{n}.$$

For each $\varepsilon_i \in \{-, +\}$ denote by $\mathcal{L}_{\varepsilon_i}$ the set $\mathcal{L}_{\varepsilon_i} := \{n \in \mathbb{N} : \lambda(n) = \varepsilon_i 1\}$. We make use of the following equivalence.

Lemma 1.13. $\sum_{n \leq x} \lambda(n) = o(x)$ if and only if $d(\mathcal{L}_+) = d(\mathcal{L}_-) = \frac{1}{2}$.

Proof. This statement is easily realized by the fact that

$$d(\mathcal{L}_+) - d(\mathcal{L}_-) = \lim_{N \rightarrow \infty} \frac{\mathcal{L}_+(N) - \mathcal{L}_-(N)}{N} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \lambda(n). \quad (1.6)$$

If $d(\mathcal{L}_+) = d(\mathcal{L}_-) = \frac{1}{2}$, using (1.6), $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} \lambda(k) = 0$ trivially.

Conversely, if $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \lambda(n) = 0$, again appealing to (1.6), it must be the case that $d(\mathcal{L}_+) = d(\mathcal{L}_-)$. Noting that

$$\lim_{n \rightarrow \infty} \frac{\mathcal{L}_+(n) + \mathcal{L}_-(n)}{n} = 1,$$

requires the common value of $d(\mathcal{L}_+)$ and $d(\mathcal{L}_-)$ to be $\frac{1}{2}$. □

To establish Theorem 1.12, we prove an amazingly simple link between the concept of density in elementary number theory and the asymptotic behavior of certain zeta functions.

1.2.2 A density–residue theorem

For A a subset of \mathbb{N} we define the *zeta function associated to A* , denoted $\zeta_A(s)$, as

$$\zeta_A(s) := \sum_{a \in A} \frac{1}{a^s},$$

where s is taken to be in the half plane of convergence. Using these definitions we have the following theorem.

Theorem 1.14. *Let A be a subset of \mathbb{N} and $s = 1$ be the right-most pole of $\zeta_A(s)$. If $s = 1$ is a simple pole of $\zeta_A(s)$ and $\zeta_A(s)$ can be analytically continued to a region R which contains the half-plane $\Re(s) \geq 1$ ($s \neq 1$), then $d(A)$ exists and is equal to $\operatorname{Res}_{s=1} \{\zeta_A(s)\}$.*

Proof. Following [7, p. 243 Lemma 4], we define

$$F(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s \frac{ds}{s} = \begin{cases} 1 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{if } 0 < x < 1. \end{cases} \quad (c > 0)$$

Sending $x \mapsto x/a$ and summing over all $a \in A$ with $a \leq x$ gives, for $\varepsilon > 0$ some arbitrarily small quantity, we have

$$A(x) = \frac{1}{2\pi i} \int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \sum_{\substack{a \leq x-1 \\ a \in A}} \frac{1}{a^s} x^s \frac{ds}{s} + c \cdot F(1), \quad (1.7)$$

where $c = 1$ if $x \in A$ and $c = 0$ if $x \notin A$. In either case, clearly $c \cdot F(1) = o(x)$. Since $F(x/a) = 0$ when $x < a$, we may extend the sum in (1.7) to all of A . Hence

$$A(x) = \frac{1}{2\pi i} \int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \zeta_A(s) \cdot x^s \frac{ds}{s} + c \cdot F(1).$$

Since $\zeta_A(s)$ is analytically continuable to a region R containing $\Re(s) \geq 1$ ($s \neq 1$) and the right-most pole of $\zeta_A(s)$ is simple, and occurs at $s = 1$, we gain

$$A(x) = x \cdot \operatorname{Res}_{s=1} \{\zeta_A(s)\} + \frac{1}{2\pi i} \int_{\partial R} \zeta_A(s) \cdot x^s \frac{ds}{s} + c \cdot F(1),$$

so that

$$\frac{A(x)}{x} = \operatorname{Res}_{s=1} \{\zeta_A(s)\} + \frac{1}{x} \cdot \frac{1}{2\pi i} \int_{\partial R} \zeta_A(s) \cdot x^s \frac{ds}{s} + \frac{1}{x} \cdot c \cdot F(1), \quad (1.8)$$

where ∂R denotes the boundary of R . Since R is a region of analyticity of a function, R is open, and so R contains the right half-plane $\Re(s) \geq 1$; thus the integral in (1.8) is $o(x)$. To make this explicit, one may take the boundary of this region to be the contour

$$\mathcal{C} := \left\{ 1 - \frac{f(t)}{2} + it : t \in \mathbb{R} \right\},$$

where $f(t)$ is the distance from the point $1 + it$ to the nearest pole of $\zeta_A(s)$ in the right half-plane $\Re s < 1$. Since $\zeta_A(s)$ can be analytically continued to a region R which contains the half-plane $\Re(s) \geq 1$ ($s \neq 1$), the distance from each point on the line $\Re s = 1$ to \mathcal{C}

is necessarily positive, and hence so is the distance to ∂R . Hence we have shown that \mathcal{C} bounds a region R which contains the half-plane $\Re s \geq 1$.

Thus the limit of the right-hand side as $x \rightarrow \infty$ of (1.8) exists and is equal to $\operatorname{Res}_{s=1} \{\zeta_A(s)\}$. Hence the limit of the left-hand side of (1.8) exists and is equal to $\operatorname{Res}_{s=1} \{\zeta_A(s)\}$; that is, $d(A)$ exists and

$$d(A) = \lim_{x \rightarrow \infty} \frac{A(x)}{x} = \operatorname{Res}_{s=1} \{\zeta_A(s)\},$$

which is the desired result. \square

The proof of Theorem 1.14 is new, though the result is not. Indeed, Theorem 1.14 contains special cases of both the Wiener–Ikehara Theorem [60, 95] and the Halász–Wirsing Mean Value Theorem [54, 97], the proofs of which, in full generality, are much more involved than the special case given above.

1.2.3 Proofs of Theorems 1.11 and 1.12

Proof of Theorem 1.11. Noting that $(1 - z^2) = (1 + z)(1 - z)$, using the Euler product formula we have for $\Re s > 1$

$$\sum_{n \geq 1} \frac{\lambda(n)}{n^s} = \prod_p \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} = \prod_p \left(1 + \frac{1}{p^s}\right)^{-1} = \prod_p \frac{\left(1 - \frac{1}{p^{2s}}\right)^{-1}}{\left(1 - \frac{1}{p^s}\right)^{-1}} = \frac{\zeta(2s)}{\zeta(s)}.$$

Since $\zeta(s)$ has a pole at $s = 1$, and converges at $s = 2$, we have that

$$\lim_{s \rightarrow 1^+} \frac{\zeta(2s)}{\zeta(s)} = 0.$$

To construct an analytic continuation of $\sum_{n \geq 1} \frac{\lambda(n)}{n^s}$ to the region $\Re s \geq 1$, we define

$$Z(s) := \begin{cases} \frac{\zeta(2s)}{\zeta(s)} & \text{in the region } \Re s \geq 1, s \neq 1, \\ 0 & \text{on the line } s = 1. \end{cases}$$

Now if $Z(s)$ is analytic in the region $\Re s \geq 1$ we have found the unique analytic continuation of $\sum_{n \geq 1} \frac{\lambda(n)}{n^s}$ to this region. Note here that $Z(s)$ is analytic in the region $\Re s \geq 1$ if and only if $\zeta(s)$ is non-vanishing in this region; this gives the equivalence of (i) and (ii) of Theorem 1.11. \square

Proof of Theorem 1.12. Consider the function $\zeta_{\mathcal{L}_+}(s) = \sum_{n \in \mathcal{L}_+} n^{-s}$. For $\Re(s) > 1$ we have $\zeta_{\mathcal{L}_+}(s) = \sum_{n \in \mathbb{N}} l(n)n^{-s}$ where $l : \mathbb{N} \rightarrow \{0, 1\}$ is defined by

$$l(i) := \frac{1 + \lambda(i)}{2}.$$

Also for $\Re(s) > 1$,

$$\zeta_{\mathcal{L}_+}(s) = \frac{1}{2} \sum_{n \in \mathbb{N}} \frac{1 + \lambda(n)}{n^s} = \frac{1}{2} \left(\zeta(s) + \frac{\zeta(2s)}{\zeta(s)} \right) = \frac{\zeta(s)^2 + \zeta(2s)}{2 \cdot \zeta(s)}. \quad (1.9)$$

Since $\zeta(s)$ is analytically continuable to a meromorphic function on all of \mathbb{C} , the relation in (1.9) implies the same for $\zeta_{\mathcal{L}_+}(s)$. Again using (1.9), since $\zeta(s)$ is non-zero in the region $\Re s \geq 1$, the function $\zeta_{\mathcal{L}_+}(s)$ has no poles in the region $\Re s \geq 1$, except at $s = 1$. Furthermore,

$$\operatorname{Res}_{s=1} \{\zeta_{\mathcal{L}_+}(s)\} = \frac{1}{2} \cdot \operatorname{Res}_{s=1} \{\zeta(s)\}.$$

Hence $\zeta_{\mathcal{L}_+}(s)$ is analytic at $s = 1 + it$ for all real $t \neq 0$, since at these s , $\zeta(s)$ is non-zero and analytic. Thus, the existence of a meromorphic continuation of $\zeta_{\mathcal{L}_+}(s)$ to all of \mathbb{C} , implies the existence of a region of analyticity of $\zeta_{\mathcal{L}_+}(s)$ containing the right half-plane $\Re s \geq 1$ with the exception of the pole at $s = 1$.

Using (1.9), the definition of $Z(s)$ in the proof of Theorem 1.11, and the region R described in the preceding paragraph, the function $\zeta_{\mathcal{L}_+}(s)$ satisfies all of the assumptions of Theorem 1.14. Applying Theorem 1.14 gives both the existence of $d(\mathcal{L}_+)$ and the value

$$d(\mathcal{L}_+) = \operatorname{Res}_{s=1} \{\zeta_{\mathcal{L}_+}(s)\} = \frac{1}{2} \cdot \operatorname{Res}_{s=1} \{\zeta(s)\} = \frac{1}{2}.$$

An application of Lemma 1.13 yields $\sum_{n \leq x} \lambda(n) = o(x)$. □

Chapter 2

Generalized Liouville functions

This chapter contains results which were found in collaboration with Peter Borwein and Stephen K.K. Choi (see [13] for details).

2.1 Introduction

Let $\Omega(n)$ be the number of distinct prime factors in n (with multiple factors counted multiply). Recall that the Liouville λ -function is defined by

$$\lambda(n) := (-1)^{\Omega(n)}.$$

So $\lambda(1) = \lambda(4) = \lambda(6) = \lambda(9) = \lambda(10) = 1$ and $\lambda(2) = \lambda(5) = \lambda(7) = \lambda(8) = -1$. In particular, $\lambda(p) = -1$ for any prime p . It is well-known [55, Section 22.10] that Ω is completely additive, i.e., $\Omega(mn) = \Omega(m) + \Omega(n)$ for any m and n and hence λ is completely multiplicative, i.e., $\lambda(mn) = \lambda(m)\lambda(n)$ for all $m, n \in \mathbb{N}$. It is interesting to note that on the set of square-free positive integers $\lambda(n) = \mu(n)$, where μ is the Möbius function. In this respect, the Liouville λ -function can be thought of as a modification of the Möbius function.

Similar to the Möbius function, many investigations surrounding the λ -function concern the summatory function of initial values of λ ; that is, the sum

$$L(x) := \sum_{n \leq x} \lambda(n).$$

Historically, this function has been studied by many mathematicians, including Liouville, Landau, Pólya, and Turán. Recent attention to the summatory function of the Möbius

function has been given by Ng [80, 81]. Larger classes of completely multiplicative functions have been studied by Granville and Soundararajan [50, 51, 52].

One of the most important questions is that of the asymptotic order of $L(x)$; more formally, the question is to determine the smallest value of ϑ for which

$$\lim_{x \rightarrow \infty} \frac{L(x)}{x^\vartheta} = 0.$$

It is known that the value of $\vartheta = 1$ is given by the prime number theorem [65, 66] and that $\vartheta = \frac{1}{2} + \varepsilon$ for any arbitrarily small positive constant ε is equivalent to the Riemann hypothesis [14]. The value of $\frac{1}{2} + \varepsilon$ is best possible, as $\limsup_{x \rightarrow \infty} L(x)/\sqrt{x} > .061867$; see Borwein, Ferguson, and Mossinghoff [19]. Indeed, any result asserting a fixed $\vartheta \in (\frac{1}{2}, 1)$ would give an expansion of the zero-free region of the Riemann zeta function, $\zeta(s)$, to $\Re s \geq \vartheta$.

Unfortunately, a closed form for $L(x)$ is unknown. This brings us to the motivating question behind the investigation of this chapter: *are there functions similar to λ , so that the corresponding summatory function does yield a closed form?*

Throughout this investigation P denotes the set of all primes. As an analogue to the traditional λ and Ω consider the following definition.

Definition 2.1. Define the *Liouville function* for $A \subset P$ by

$$\lambda_A(n) = (-1)^{\Omega_A(n)}$$

where $\Omega_A(n)$ is the number of prime factors of n , counting multiplicity, coming from A . The set of all of these functions is denoted $\mathcal{F}(\{-1, 1\})$; this notation is introduced by Granville and Soundararajan in [51].

Alternatively, one can define λ_A as the completely multiplicative function with $\lambda_A(p) = -1$ for each prime $p \in A$ and $\lambda_A(p) = 1$ for all $p \notin A$. Every completely multiplicative function taking only ± 1 values is built this way. Also, denote

$$L_A(x) := \sum_{n \leq x} \lambda_A(n) \quad \text{and} \quad R_A := \lim_{n \rightarrow \infty} \frac{L_A(x)}{n}.$$

In this chapter, we first consider questions regarding the properties of the function λ_A by studying the limit R_A . The structure of R_A is determined and it is shown that for each $\alpha \in [0, 1]$ there is a subset A of primes such that $R_A = \alpha$. The rest of this chapter considers an extended investigation on those functions in $\mathcal{F}(\{-1, 1\})$ that are character-like in nature, meaning that there is a real Dirichlet character χ such that $\lambda_A(n) = \chi(n)$ whenever $\chi(n) \neq 0$. Within the course of discussion, the ratio $\varphi(n)/\sigma(n)$ is considered.

2.2 Properties of $L_A(x)$

Define the *generalized Liouville sequence* as

$$\mathfrak{L}_A := (\lambda_A(1), \lambda_A(2), \dots).$$

Theorem 2.2. *If $A \neq \emptyset$, then the sequence \mathfrak{L}_A is not eventually periodic.*

Proof. Towards a contradiction, suppose that \mathfrak{L}_A is eventually periodic, say the sequence is periodic after the M -th term and has period k . Now there is an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $nk > M$. Since $A \neq \emptyset$, pick $p \in A$. Then

$$\lambda_A(pnk) = \lambda_A(p) \cdot \lambda_A(nk) = -\lambda_A(nk).$$

But $pnk \equiv nk \pmod{k}$, a contradiction to the eventual k -periodicity of \mathfrak{L}_A . \square

Corollary 2.3. *If $A \subset P$ is nonempty, then λ_A is not a Dirichlet character.*

Proof. This is a direct consequence of the non-periodicity of \mathfrak{L}_A . \square

To get more acquainted with the sequence \mathfrak{L}_A , we study the partial sums $L_A(x)$ of \mathfrak{L}_A , and to study these, we consider the Dirichlet series with coefficients $\lambda_A(n)$.

Starting with singleton sets $\{p\}$ of the primes, a nice relation becomes apparent; for $\Re s > 1$ we have

$$\frac{(1 - p^{-s})}{(1 + p^{-s})} \zeta(s) = \sum_{n \geq 1} \frac{\lambda_{\{p\}}(n)}{n^s}, \quad (2.1)$$

and for sets $\{p, q\}$, the following identity holds:

$$\frac{(1 - p^{-s})(1 - q^{-s})}{(1 + p^{-s})(1 + q^{-s})} \zeta(s) = \sum_{n \geq 1} \frac{\lambda_{\{p, q\}}(n)}{n^s}. \quad (2.2)$$

Since λ_A is completely multiplicative, for any subset A of primes, for $\Re s > 1$ we have

$$\begin{aligned} \mathcal{L}_A(s) &:= \sum_{n \geq 1} \frac{\lambda_A(n)}{n^s} = \prod_p \left(\sum_{l \geq 0} \frac{\lambda_A(p^l)}{p^{ls}} \right) \\ &= \prod_{p \in A} \left(\sum_{l \geq 0} \frac{(-1)^l}{p^{ls}} \right) \prod_{p \notin A} \left(\sum_{l \geq 0} \frac{1}{p^{ls}} \right) = \prod_{p \in A} \left(\frac{1}{1 + \frac{1}{p^s}} \right) \prod_{p \notin A} \left(\frac{1}{1 - \frac{1}{p^s}} \right) \\ &= \zeta(s) \prod_{p \in A} \left(\frac{1 - p^{-s}}{1 + p^{-s}} \right). \end{aligned} \quad (2.3)$$

This relation leads us to our next theorem, but first let us recall a piece of notation from the last section.

Definition 2.4. For $A \subset P$ denote

$$R_A := \lim_{n \rightarrow \infty} \frac{\lambda_A(1) + \lambda_A(2) + \dots + \lambda_A(n)}{n}.$$

The existence of the limit R_A is guaranteed by Wirsing's Theorem. In fact, Wirsing [97] showed more generally that every real multiplicative function f with $|f(n)| \leq 1$ has a mean value, i.e, the limit

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n)$$

exists. Furthermore, Wintner [96] showed that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n) = \prod_p \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots \right) \left(1 - \frac{1}{p} \right) \neq 0$$

if and only if $\sum_p |1 - f(p)|/p$ converges. Otherwise the mean value is zero. This gives the following theorem.

Theorem 2.5. For the completely multiplicative function $\lambda_A(n)$, the limit R_A exists and

$$R_A = \begin{cases} \prod_{p \in A} \frac{p-1}{p+1} & \text{if } \sum_{p \in A} p^{-1} < \infty, \\ 0 & \text{otherwise.} \end{cases} \quad (2.4)$$

Example 2.6. For any prime p , $R_{\{p\}} = \frac{p-1}{p+1}$.

Let us make some notational comments. Denote by $\mathcal{P}(P)$ the power set of the set of primes. Note that

$$\frac{p-1}{p+1} = 1 - \frac{2}{p+1}.$$

Recall from above that $R : \mathcal{P}(P) \rightarrow \mathbb{R}$ is defined by

$$R_A := \prod_{p \in A} \left(1 - \frac{2}{p+1} \right).$$

It is immediate that R is bounded above by 1 and below by 0, so that we need only consider that $R : \mathcal{P}(P) \rightarrow [0, 1]$. It is also immediate that $R_\emptyset = 1$ and $R_P = 0$.

Remark 2.7. For $n \in \mathbb{N}$, let p_n be the smallest prime larger than n^3 ; i.e. $p_n := \min_{q > n^3} \{q \in P\}$. Since there is always a prime in the interval $(x, x + x^{5/8}]$ (see [61]), we have $p_{n+1} > p_n$ for all $n \in \mathbb{N}$. Let

$$K := \{p_n : n \in \mathbb{N}\} = \{11, 29, 67, 127, 223, 347, 521, 733, 1009, 1361, \dots\}.$$

Note that

$$\frac{p_n - 1}{p_n + 1} > \frac{n^3 - 1}{n^3 + 1},$$

so that

$$R_K = \prod_{p \in K} \left(\frac{p-1}{p+1} \right) > \prod_{n \geq 2} \left(\frac{n^3 - 1}{n^3 + 1} \right) = \frac{2}{3}.$$

Also $R_K < (11 - 1)/(11 + 1) = 5/6$, so that

$$\frac{2}{3} < R_K < \frac{5}{6},$$

and $R_K \in (0, 1)$.

There are some very interesting and important examples of sets of primes A for which $R_A = 0$. Indeed, results of von Mangoldt [94] and Landau [65, 66] give the following equivalence.

Theorem 2.8. *The prime number theorem is equivalent to $R_P = 0$.*

We may be a bit more specific regarding the values of R_A , for $A \in \mathcal{P}(P)$. For each $\alpha \in (0, 1)$, there is a set of primes A such that

$$R_A = \prod_{p \in A} \left(\frac{p-1}{p+1} \right) = \alpha.$$

This result is a special case of some general theorems of Granville and Soundararajan [51].

Theorem 2.9 (Granville and Soundararajan [51]). *The function $R : \mathcal{P}(P) \rightarrow [0, 1]$ is surjective. That is, for each $\alpha \in [0, 1]$ there is a set of primes A such that $R_A = \alpha$.*

Proof. This follows from Corollary 2 and Theorem 4 (ii) of [51] with $S = \{-1, 1\}$, though in this special case, a much more elementary argument can yield the result.

To this end, not first that $R_P = 0$ and $R_\emptyset = 1$. To prove the statement for the remainder of the values, let $\alpha \in (0, 1)$. Then since

$$\lim_{p \rightarrow \infty} R_{\{p\}} = \lim_{p \rightarrow \infty} \left(1 - \frac{2}{p+1} \right) = 1,$$

there is a minimal prime q_1 such that

$$R_{\{q_1\}} = \left(1 - \frac{2}{q_1 + 1}\right) > \alpha,$$

i.e.,

$$\frac{1}{\alpha} \cdot R_{\{q_1\}} = \frac{1}{\alpha} \left(1 - \frac{2}{q_1 + 1}\right) > 1.$$

Similarly, for each $N \in \mathbb{N}$, we may continue in the same fashion, choosing $q_i > q_{i-1}$ (for $i = 2 \dots N$) minimally, we have

$$\frac{1}{\alpha} \cdot R_{\{q_1, q_2, \dots, q_N\}} = \frac{1}{\alpha} \prod_{i=1}^N \left(1 - \frac{2}{q_i + 1}\right) > 1.$$

Now consider

$$\lim_{N \rightarrow \infty} \frac{1}{\alpha} \cdot R_{\{q_1, q_2, \dots, q_N\}} = \frac{1}{\alpha} \prod_{i \geq 1} \left(1 - \frac{2}{q_i + 1}\right),$$

where the q_i are chosen as before. Denote $A = \{q_i : i \in \mathbb{N}\}$. We know that

$$\frac{1}{\alpha} \cdot R_A = \frac{1}{\alpha} \prod_{i \geq 1} \left(1 - \frac{2}{q_i + 1}\right) \geq 1.$$

We claim that $R_A = \alpha$. To this end, let us suppose to the contrary that

$$\frac{1}{\alpha} \cdot R_A = \frac{1}{\alpha} \prod_{i \geq 1} \left(1 - \frac{2}{q_i + 1}\right) > 1.$$

Note that $P \setminus A$ is infinite (here P is the set of all primes). As earlier, since

$$\lim_{\substack{p \rightarrow \infty \\ p \in A \setminus P}} R_{\{p\}} = \lim_{p \rightarrow \infty} \left(1 - \frac{2}{p + 1}\right) = 1,$$

there is a minimal prime $q \in A \setminus P$ such that

$$\frac{1}{\alpha} \cdot R_A \cdot R_{\{q\}} = \frac{1}{\alpha} \left[\prod_{i \geq 1} \left(1 - \frac{2}{q_i + 1}\right) \right] \cdot \left(1 - \frac{2}{q + 1}\right) > 1.$$

Since q is a prime and $q \notin A$, there is an $i \in \mathbb{N}$ with $q_i < q < q_{i+1}$. This contradicts that q_{i+1} was a minimal choice. Hence

$$\frac{1}{\alpha} \cdot R_A = \frac{1}{\alpha} \prod_{i \geq 1} \left(1 - \frac{2}{q_i + 1}\right) = 1,$$

and there is a set A of primes such that $R_A = \alpha$. □

In fact, let S denote a subset of the unit disk and let $\mathcal{F}(S)$ be the class of totally multiplicative functions such that $f(p) \in S$ for all primes p . Granville and Soundararajan [51] prove very general results concerning both the Euler product spectrum $\Gamma_\theta(S)$ and the spectrum $\Gamma(S)$ of the class $\mathcal{F}(S)$.

The following theorem gives asymptotic formulas for the mean value of λ_A if certain conditions on the density of A in P are assumed.

Theorem 2.10. *Let A be a subset of primes and suppose*

$$\sum_{\substack{p \leq x \\ p \in A}} \frac{\log p}{p} = \frac{1 - \kappa}{2} \log x + O(1) \quad (2.5)$$

with $-1 \leq \kappa \leq 1$.

If $0 < \kappa \leq 1$, then we have

$$\sum_{n \leq x} \frac{\lambda_A(n)}{n} = c_\kappa (\log x)^\kappa + O(1)$$

and

$$\sum_{n \leq x} \lambda_A(n) = (1 + o(1)) c_\kappa \kappa x (\log x)^{\kappa-1},$$

where

$$c_\kappa = \frac{1}{\Gamma(\kappa + 1)} \prod_p \left(1 - \frac{1}{p}\right)^\kappa \left(1 - \frac{\lambda_A(p)}{p}\right)^{-1}. \quad (2.6)$$

In particular,

$$R_A = \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \lambda_A(n) = \begin{cases} c_1 = \prod_{p \in A} \left(\frac{p-1}{p+1}\right) & \text{if } \kappa = 1, \\ 0 & \text{if } 0 < \kappa < 1. \end{cases}$$

Furthermore, $\mathcal{L}_A(s) = \sum_{n \geq 1} \frac{\lambda_A(n)}{n^s}$ has a pole of order κ at $s = 1$ with residue $c_\kappa \Gamma(\kappa + 1)$; that is,

$$\mathcal{L}_A(s) = \frac{c_\kappa \Gamma(\kappa + 1)}{(s - 1)^\kappa} + \psi(s), \quad \Re s > 1,$$

for some function $\psi(s)$ analytic in the region $\Re s \geq 0$. If $-1 \leq \kappa < 0$, then $\mathcal{L}_A(s)$ has zero of order $-\kappa$ at $s = 1$ and

$$\mathcal{L}_A(s) = \frac{\zeta(2s)}{c_{-\kappa} \Gamma(-\kappa + 1)} (s - 1)^{-\kappa} (1 + \varphi(s))$$

for some function $\varphi(s)$ analytic in the region $\Re s \geq 1$ and hence

$$\mathcal{L}_A(1) = \sum_{n \geq 1} \frac{\lambda_A(n)}{n} = 0$$

and

$$R_A = \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \lambda_A(n) = 0.$$

If $\kappa = 0$, then $\mathcal{L}_A(s)$ has neither pole nor zero at $s = 1$. In particular, we have

$$\sum_{n \geq 1} \frac{\lambda_A(n)}{n} = \alpha \neq 0$$

and

$$R_A = \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \lambda_A(n) = 0.$$

The proof of Theorem 2.10 requires the following result.

Theorem 2.11 (Wirsing [97]). *Suppose f is a completely multiplicative function which satisfies*

$$\sum_{n \leq x} \Lambda(n) f(n) = \kappa \log x + O(1) \tag{2.7}$$

and

$$\sum_{n \leq x} |f(n)| \ll \log x \tag{2.8}$$

with $0 \leq \kappa \leq 1$ where $\Lambda(n)$ is the von Mangoldt function. Then we have

$$\sum_{n \leq x} f(n) = c_f (\log x)^\kappa + O(1) \tag{2.9}$$

where

$$c_f := \frac{1}{\Gamma(\kappa + 1)} \prod_p \left(1 - \frac{1}{p}\right)^\kappa \left(\frac{1}{1 - f(p)}\right) \tag{2.10}$$

where $\Gamma(\kappa)$ is the Gamma function.

Proof of Theorem 2.10. Suppose first that $0 < \kappa \leq 1$. We choose $f(n) = \frac{\lambda_A(n)}{n}$ in Wirsing's Theorem. Since

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \sum_{p \leq x} \frac{\log p}{p} + O(1) = \log x + O(1),$$

we have

$$\begin{aligned}
 \sum_{n \leq x} \Lambda(n) \frac{\lambda_A(n)}{n} &= \sum_{p \leq x} \log p \frac{\lambda_A(p)}{p} + O\left(\sum_{\substack{p^l \leq x \\ l \geq 2}} \frac{\log p}{p^l}\right) \\
 &= \sum_{p \leq x} \log p \frac{\lambda_A(p)}{p} + O\left(\sum_{n \leq x} \frac{\Lambda(n)}{n} - \sum_{p \leq x} \frac{\log p}{p}\right) \\
 &= \sum_{p \leq x} \log p \frac{\lambda_A(p)}{p} + O(1).
 \end{aligned}$$

On the other hand, from (2.5) we have

$$\begin{aligned}
 \sum_{p \leq x} \log p \frac{\lambda_A(p)}{p} &= \sum_{p \leq x} \frac{\log p}{p} - 2 \sum_{\substack{p \leq x \\ p \in A}} \frac{\log p}{p} \\
 &= \kappa \log x + O(1).
 \end{aligned}$$

Hence

$$\sum_{n \leq x} \Lambda(n) \frac{\lambda_A(n)}{n} = \kappa \log x + O(1)$$

and condition (2.7) is satisfied.

It then follows from (2.9) and (2.6) that

$$\sum_{n \leq x} \frac{\lambda_A(n)}{n} = c_\kappa (\log x)^\kappa + O(1).$$

From (2.5),

$$\begin{aligned}
 \mathcal{L}_A(s+1) &= \sum_{n \geq 1} \frac{\lambda_A(n)}{n^{s+1}} = \int_1^\infty y^{-s} d \sum_{n \leq y} \frac{\lambda_A(n)}{n} \\
 &= \int_1^\infty y^{-s} d(c_\kappa (\log y)^\kappa + O(1)) \\
 &= c_\kappa \kappa \int_1^\infty \frac{(\log y)^{\kappa-1}}{y^{s+1}} dy + \int_1^\infty y^{-s} dO(1) \\
 &= c_\kappa \Gamma(\kappa+1) s^{-\kappa} + \psi(s)
 \end{aligned}$$

for $\Re s > 0$, because

$$\int_1^\infty \frac{(\log y)^{\kappa-1}}{y^{s+1}} dy = \Gamma(\kappa) s^{-\kappa}.$$

Here $\psi(s)$ is an analytic function on $\Re s \geq 0$.

Therefore, $\mathcal{L}_A(s)$ has a pole at $s = 1$ of order $0 < \kappa \leq 1$. Now from a generalization of the Wiener-Ikehara theorem [9, Theorem 7.7], we have

$$\sum_{n \leq x} \lambda_A(n) = (1 + o(1)) c_\kappa \kappa x (\log x)^{\kappa-1}$$

and hence

$$R_A = \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \lambda_A(n) = \begin{cases} c_1 & \text{if } \kappa = 1, \\ 0 & \text{if } 0 < \kappa < 1, \end{cases}$$

where

$$c_1 = \prod_p \left(1 - \frac{1}{p}\right) \left(1 - \frac{\lambda_A(p)}{p}\right)^{-1} = \prod_{p \in A} \left(\frac{1 - p^{-1}}{1 + p^{-1}}\right).$$

Denote the complement of A by \bar{A} . If $-1 \leq \kappa < 0$, then we have

$$\begin{aligned} \mathcal{L}_{\bar{A}}(s) &= \sum_{n \geq 1} \frac{\lambda_{\bar{A}}(n)}{n^s} = \zeta(s) \prod_{p \notin A} \left(\frac{1 - p^{-s}}{1 + p^{-s}}\right) \\ &= \frac{\zeta(2s)}{\zeta(s)} \prod_{p \in A} \left(\frac{1 + p^{-s}}{1 - p^{-s}}\right) = \frac{\zeta(2s)}{\mathcal{L}_A(s)} \end{aligned}$$

for $\Re s > 1$. Hence, for $\Re s > 1$, we have

$$\mathcal{L}_{\bar{A}}(s) \mathcal{L}_A(s) = \zeta(2s). \quad (2.11)$$

From (2.5), we have

$$\sum_{\substack{p \leq x \\ p \notin A}} \frac{\log p}{p} = \sum_{p \leq x} \frac{\log p}{p} - \sum_{\substack{p \leq x \\ p \in A}} \frac{\log p}{p} = \frac{1 + \kappa}{2} \log x + O(1)$$

and

$$\sum_{n \leq x} \Lambda(n) \frac{\lambda_{\bar{A}}(n)}{n} = -\kappa \log x + O(1).$$

We then apply the above case to $\mathcal{L}_{\bar{A}}(s)$ and deduce that $\mathcal{L}_{\bar{A}}(s)$ has a pole at $s = 1$ of order $-\kappa$, then in view of (2.11), $\mathcal{L}_A(s)$ has a zero at $s = 1$ of order $-\kappa$; that is,

$$\mathcal{L}_A(s) = \frac{\zeta(2s)}{c_{-\kappa} \Gamma(-\kappa + 1)} (s - 1)^{-\kappa} (1 + \varphi(s))$$

for some function $\varphi(s)$ analytic on the region $\Re s \geq 1$. In particular, we have

$$\mathcal{L}_A(1) = \sum_{n \geq 1} \frac{\lambda_A(n)}{n} = 0. \quad (2.12)$$

This completes the proof of Theorem 2.10. \square

Recall that Theorem 2.9 tells us that any $\alpha \in [0, 1]$ is a mean value of a function in $\mathcal{F}(\{-1, 1\})$. The functions in $\mathcal{F}(\{-1, 1\})$ can be put into two natural classes: those with mean value 0 and those with positive mean value.

Asymptotically, those functions with mean value zero are more interesting, and it is in this class which the Liouville λ -function resides, and in that which concerns the prime number theorem and the Riemann hypothesis. We consider an extended example of such functions in Section 2.4. Before this consideration, we ask some questions about those functions $f \in \mathcal{F}(\{-1, 1, \})$ with positive mean value.

2.3 One question twice

It is obvious that if $\alpha \notin \mathbb{Q}$, then $R_A \neq \alpha$ for any finite set $A \subset P$. We also know that if $A \subset P$ is finite, then $R_A \in \mathbb{Q}$.

Question 2.12. For $\alpha \in \mathbb{Q}$ is there a finite subset A of P , such that $R_A = \alpha$?

The above question can be posed in a more interesting fashion. Indeed, note that for any finite set of primes A , we have that

$$R_A = \prod_{p \in A} \frac{p-1}{p+1} = \prod_{p \in A} \frac{\varphi(p)}{\sigma(p)} = \frac{\varphi(z)}{\sigma(z)}$$

where $z = \prod_{p \in A} p$, φ is Euler's totient function, and σ is the sum of divisors function. Alternatively, we may view the finite set of primes A as determined by the square-free integer z . In fact, the function f from the set of square-free integers to the set of finite subsets of primes, defined by

$$f(z) = f(p_1 p_2 \cdots p_r) = \{p_1, p_2, \dots, p_r\}, \quad (z = p_1 p_2 \cdots p_r)$$

is bijective, giving a one-to-one correspondence between these two sets.

In this terminology, we ask the question as:

Question 2.13. Is the image of $\varphi(z)/\sigma(z) : \{\text{square-free integers}\} \rightarrow \mathbb{Q} \cap (0, 1)$ a surjection?

That is, for every rational $q \in (0, 1)$, is there a square-free integer z such that $\frac{\varphi(z)}{\sigma(z)} = q$? As a start, we have Theorem 2.9, which gives a nice corollary.

Corollary 2.14. *If S is the set of square-free integers, then*

$$\left\{ x \in \mathbb{R} : x = \lim_{\substack{k \rightarrow \infty \\ \{n_k\} \subset S}} \frac{\varphi(n_k)}{\sigma(n_k)} \right\} = [0, 1];$$

that is, the set $\{\varphi(s)/\sigma(s) : s \in S\}$ is dense in $[0, 1]$.

Proof. Let $\alpha \in [0, 1]$ and A be a subset of primes for which $R_A = \alpha$. If A is finite we are done, so suppose A is infinite. Write

$$A = \{a_1, a_2, a_3, \dots\}$$

where $a_i < a_{i+1}$ for $i = 1, 2, 3, \dots$ and define $n_k = \prod_{i=1}^k a_i$. The sequence (n_k) satisfies the needed limit. \square

2.4 The functions $\lambda_p(n)$

We now turn our attention to those functions $\mathcal{F}(\{-1, 1\})$ with mean value 0; in particular, we wish to examine functions for which a sort of Riemann hypothesis holds: functions for which $\mathcal{L}_A(s) = \sum_{n \geq 1} \frac{\lambda_A(n)}{n^s}$ has a large zero-free region. These are functions for which $\sum_{n \leq x} \lambda_A(n)$ grows slowly.

To this end, let p be a prime number. Recall that the Legendre symbol modulo p is defined as

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } q \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } q \text{ is a quadratic non-residue modulo } p, \\ 0 & \text{if } q \equiv 0 \pmod{p}. \end{cases}$$

Here q is a quadratic residue modulo p provided $q \equiv x^2 \pmod{p}$ for some $x \not\equiv 0 \pmod{p}$.

Define the function $\Omega_p(n)$ to be the number of prime factors q , of n with $\left(\frac{q}{p}\right) = -1$; that is,

$$\Omega_p(n) = \#\left\{q : q \text{ is a prime, } q \mid n, \text{ and } \left(\frac{q}{p}\right) = -1\right\}.$$

Definition 2.15. The *Liouville function for quadratic non-residues modulo p* is defined as

$$\lambda_p(n) := (-1)^{\Omega_p(n)}.$$

The function $\Omega_p(n)$ is completely additive since it counts primes with multiplicities. Thus $\lambda_p(n)$ is completely multiplicative.

Lemma 2.16. *The function $\lambda_p(n)$ is the unique completely multiplicative function defined by $\lambda_p(p) = 1$, and for primes $q \neq p$ by*

$$\lambda_p(q) = \left(\frac{q}{p}\right).$$

Proof. Let q be a prime with $q \mid n$. Now $\Omega_p(q) = 0$ or 1 depending on whether $\left(\frac{q}{p}\right) = 1$ or -1 , respectively. If $\left(\frac{q}{p}\right) = 1$, then $\Omega_p(q) = 0$, and so $\lambda_p(q) = 1$.

On the other hand, if $\left(\frac{q}{p}\right) = -1$, then $\Omega_p(q) = 1$, and so $\lambda_p(q) = -1$. Note that using the given definition $\lambda_p(p) = \left(\frac{p}{p}\right) = 1$, so that in either case, we have

$$\lambda_p(q) = \left(\frac{q}{p}\right).$$

□

Hence if $n = p^k m$ with $p \nmid m$, then we have

$$\lambda_p(p^k m) = \left(\frac{m}{p}\right). \quad (2.13)$$

Similarly, we may define the function $\Omega'_p(n)$ to be the number of prime factors q of n with $\left(\frac{q}{p}\right) = 1$; that is,

$$\Omega'_p(n) = \#\left\{q : q \text{ is a prime, } q \mid n, \text{ and } \left(\frac{q}{p}\right) = 1\right\}.$$

Analogous to Lemma 2.16 we have the following lemma for $\lambda'_p(n)$ and theorem relating these two functions to the traditional Liouville λ -function.

Lemma 2.17. *The function $\lambda'_p(n)$ is the unique completely multiplicative function defined by $\lambda'_p(p) = 1$ and for primes $q \neq p$, as*

$$\lambda'_p(q) = -\left(\frac{q}{p}\right).$$

Theorem 2.18. *If $\lambda(n)$ is the standard Liouville λ -function, then*

$$\lambda(n) = (-1)^k \cdot \lambda_p(n) \cdot \lambda'_p(n)$$

where $p^k \parallel n$, i.e., $p^k \mid n$ and $p^{k+1} \nmid n$.

Proof. It is clear that the theorem is true for $n = 1$. Since all functions involved are completely multiplicative, it suffices to show the equivalence for all primes. Note that $\lambda(q) = -1$ for any prime q . Now if $n = p$, then $k = 1$ and

$$(-1)^1 \cdot \lambda_p(p) \cdot \lambda'_p(p) = (-1) \cdot (1) \cdot (1) = -1 = \lambda(p).$$

If $n = q \neq p$, then

$$(-1)^0 \cdot \lambda_p(q) \cdot \lambda'_p(q) = \left(\frac{q}{p}\right) \cdot \left(-\left(\frac{q}{p}\right)\right) = -\left(\frac{q^2}{p}\right) = -1 = \lambda(q),$$

and so the theorem is proved. \square

To mirror the relationship between L and λ , denote by $L_p(n)$, the summatory function of $\lambda_p(n)$; that is, define

$$L_p(n) := \sum_{k=1}^n \lambda_p(k).$$

It is quite immediate that $L_p(n)$ is not positive for all n and p . To find an example we need only look at the first few primes. For $p = 5$ and $n = 3$, we have

$$L_5(3) = \lambda_5(1) + \lambda_5(2) + \lambda_5(3) = 1 - 1 - 1 = -1 < 0.$$

Indeed, the next few theorems are sufficient to show that there is a positive proportion (at least $1/2$) of the primes for which $L_p(n) < 0$ for some $n \in \mathbb{N}$. For the traditional $L(n)$, it was conjectured by Pólya that $L(n) \geq 0$ for all n , though this was proven to be a non-trivial statement and ultimately false (see Haselgrove [57]).

Theorem 2.19. *Let*

$$n = a_0 + a_1p + a_2p^2 + \dots + a_kp^k$$

be the base p expansion of n , where $a_j \in \{0, 1, 2, \dots, p-1\}$. Then we have

$$L_p(n) := \sum_{l=1}^n \lambda_p(l) = \sum_{l=1}^{a_0} \lambda_p(l) + \sum_{l=1}^{a_1} \lambda_p(l) + \dots + \sum_{l=1}^{a_k} \lambda_p(l). \quad (2.14)$$

Here the sum over l is regarded as empty if $a_j = 0$.

Instead of giving a proof of Theorem 2.19 in this specific form, we will prove a more general result to which Theorem 2.19 is a direct corollary. Let χ be a non-principal Dirichlet

character modulo p and for any prime q let

$$f(q) := \begin{cases} 1 & \text{if } q = pq, \\ \chi(q) & \text{if } q \neq p. \end{cases} \quad (2.15)$$

We extend f to be a completely multiplicative function and get

$$f(p^l m) = \chi(m) \quad (2.16)$$

for $l \geq 0$ and $p \nmid m$.

Definition 2.20. Define $N(n, l)$ to be the number of times that l occurs in the base p expansion of n .

Theorem 2.21. For $N(n, l)$ as above

$$\sum_{j=1}^n f(j) = \sum_{l=0}^{p-1} N(n, l) \left(\sum_{m \leq l} \chi(m) \right).$$

Proof. We write the base p expansion of n as

$$n = a_0 + a_1 p + a_2 p^2 + \dots + a_k p^k \quad (2.17)$$

where $0 \leq a_j \leq p-1$. We then observe that, by writing $j = p^l m$ with $p \nmid m$,

$$\sum_{j=1}^n f(j) = \sum_{l=0}^k \sum_{\substack{j=1 \\ p^l \parallel j}}^n f(j) = \sum_{l=0}^k \sum_{\substack{m \leq n/p^l \\ (m,p)=1}} f(p^l m).$$

For simplicity, we write

$$A := a_0 + a_1 p + \dots + a_l p^l \quad \text{and} \quad B := a_{l+1} + a_{l+2} p + \dots + a_k p^{k-l-1}$$

so that $n = A + B p^{l+1}$ in (2.17). It now follows from (2.16) and (2.17) that

$$\sum_{j=1}^n f(j) = \sum_{l=0}^k \sum_{\substack{m \leq n/p^l \\ (m,p)=1}} \chi(m) = \sum_{l=0}^k \sum_{m \leq A/p^l + B p} \chi(m) = \sum_{l=0}^k \sum_{m \leq A/p^l} \chi(m)$$

because $\chi(p) = 0$ and $\sum_{m=a+1}^{a+p} \chi(m) = 0$ for any a . Now since

$$a_l \leq A/p^l = (a_0 + a_1 p + \dots + a_l p^l)/p^l < a_l + 1,$$

we have

$$\sum_{j=1}^n f(j) = \sum_{l=0}^k \sum_{m \leq a_l} \chi(m) = \sum_{l=0}^{p-1} N(n, l) \left(\sum_{m \leq l} \chi(m) \right).$$

This proves the theorem. □

In this language, Theorem 2.19 can be stated as follows.

Corollary 2.22. *We have*

$$L_p(n) = \sum_{j=1}^n \lambda_p(j) = \sum_{l=0}^{p-1} N(n, l) \left(\sum_{m \leq l} \binom{m}{p} \right). \quad (2.18)$$

As an application of this theorem consider $p = 3$.

Application 2.23. The value of $L_3(n)$ is equal to the number of ones in the base 3 expansion of n .

Proof. Since $\binom{1}{3} = 1$ and $\binom{1}{3} + \binom{2}{3} = 0$, if $n = a_0 + a_1 3 + a_2 3^2 + \dots + a_k 3^k$ is the base 3 expansion of n , then the right-hand side of (2.14), or equivalently the right-hand side of (2.18), is equal to $N(n, 1)$. The result then follows from either of Theorem 2.19 or Corollary 2.22. □

Note that $L_3(n) = k$ for the first time when $n = 3^0 + 3^1 + 3^2 + \dots + 3^k$ and is never negative. This is in stark contrast to the traditional $L(n)$, which is often negative. Indeed, we may classify all p for which $L_p(n) \geq 0$ for all $n \in \mathbb{N}$.

Theorem 2.24. *The function $L_p(n) \geq 0$ for all n exactly for those odd primes p for which*

$$\binom{1}{p} + \binom{2}{p} + \dots + \binom{k}{p} \geq 0$$

for all $1 \leq k \leq p$.

Proof. We first observe from (2.13) that if $0 \leq r < p$, then

$$\sum_{l=1}^r \lambda_p(l) = \sum_{l=1}^r \binom{l}{p}.$$

Let $n = a_0 + a_1p + \dots + a_kp^k$ be the base p expansion of n . From Theorem 2.19,

$$\begin{aligned} \sum_{l=1}^n \lambda_p(l) &= \sum_{l=1}^{a_0} \lambda_p(l) + \sum_{l=1}^{a_1} \lambda_p(l) + \dots + \sum_{l=1}^{a_k} \lambda_p(l) \\ &= \sum_{l=1}^{a_0} \left(\frac{l}{p}\right) + \sum_{l=1}^{a_1} \left(\frac{l}{p}\right) + \dots + \sum_{l=1}^{a_k} \left(\frac{l}{p}\right) \end{aligned}$$

because all a_j are between 0 and $p - 1$. The result then follows. \square

Corollary 2.25. *For $n \in \mathbb{N}$, we have*

$$0 \leq L_3(n) \leq [\log_3 n] + 1.$$

Proof. This follows from Theorem 2.24, Application 3.21, and the fact that the number of 1's in the base three expansion of n is $\leq [\log_3 n] + 1$. \square

As a further example, let $p = 5$.

Corollary 2.26. *The value of $L_5(n)$ is equal to the number of 1's in the base 5 expansion of n minus the number of 3's in the base 5 expansion of n . Also for $n \geq 1$,*

$$|L_5(n)| \leq [\log_5 n] + 1.$$

Recall from above, that $L_3(n)$ is always nonnegative, but $L_5(n)$ isn't. Also $L_5(n) = k$ for the first time when $n = 5^0 + 5^1 + 5^2 + \dots + 5^k$ and $L_5(n) = -k$ for the first time when $n = 3 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + \dots + 3 \cdot 5^k$.

Remark 2.27. The reason for specification of the primes p in the preceding two corollaries is that, in general, it's not always the case that $|L_p(n)| \leq [\log_p n] + 1$.

We now return to our classification of primes for which $L_p(n) \geq 0$ for all $n \geq 1$.

Definition 2.28. Denote by \mathcal{L}^+ , the set of primes p for which $L_p(n) \geq 0$ for all $n \in \mathbb{N}$.

We have found by computation that the first few values in \mathcal{L}^+ are

$$\mathcal{L}^+ = (3, 7, 11, 23, 31, 47, 59, 71, 79, 83, 103, 131, 151, 167, 191, 199, 239, 251 \dots).$$

By inspection, \mathcal{L}^+ doesn't seem to contain any primes p , with $p \equiv 1 \pmod{4}$. This is not a coincidence, as demonstrated by the following theorem.

Theorem 2.29. *If $p \in \mathcal{L}^+$, then $p \equiv 3 \pmod{4}$.*

Proof. Note that if $p \equiv 1 \pmod{4}$, then

$$\left(\frac{a}{p}\right) = \left(\frac{-a}{p}\right)$$

for all $1 \leq a \leq p-1$, so that

$$\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a}{p}\right) = 0.$$

Consider the case that $\left(\frac{(p-1)/2}{p}\right) = 1$. Then

$$\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a}{p}\right) = \sum_{a=1}^{\frac{p-1}{2}-1} \left(\frac{a}{p}\right) + \left(\frac{(p-1)/2}{p}\right) = \sum_{a=1}^{\frac{p-1}{2}-1} \left(\frac{a}{p}\right) + 1,$$

so that

$$\sum_{a=1}^{\frac{p-1}{2}-1} \left(\frac{a}{p}\right) = -1 < 0.$$

On the other hand, if $\left(\frac{(p-1)/2}{p}\right) = -1$, then since $\left(\frac{(p-1)/2}{p}\right) = \left(\frac{(p-1)/2+1}{p}\right)$, we have

$$\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a}{p}\right) = \sum_{a=1}^{\frac{p-1}{2}+1} \left(\frac{a}{p}\right) - \left(\frac{(p-1)/2+1}{p}\right) = \sum_{a=1}^{\frac{p-1}{2}+1} \left(\frac{a}{p}\right) + 1,$$

so that

$$\sum_{a=1}^{\frac{p-1}{2}+1} \left(\frac{a}{p}\right) = -1 < 0. \quad \square$$

2.5 A bound for $|L_p(n)|$

Above we were able to give exact bounds on the function $|L_p(n)|$. As explained in Remark 2.27, this is not always possible, though an asymptotic bound is easily attained with a few preliminary results.

Lemma 2.30. *For all $r, n \in \mathbb{N}$ we have $L_p(p^r n) = L_p(n)$.*

Proof. For $i = 1, \dots, p-1$ and $k \in \mathbb{N}$, $\lambda_p(kp+i) = \lambda_p(i)$. For $k \in \mathbb{N}$, this relation immediately gives that $L_p(p(k+1)-1) - L_p(pk) = 0$, since $L_p(p-1) = 0$. Thus

$$L_p(p^r n) = \sum_{k=1}^{p^r n} \lambda_p(k) = \sum_{k=1}^{p^{r-1} n} \lambda_p(pk) = \sum_{k=1}^{p^{r-1} n} \lambda_p(p) \lambda_p(k) = \sum_{k=1}^{p^{r-1} n} \lambda_p(k) = L_p(p^{r-1} n).$$

The lemma follows immediately. \square

Theorem 2.31. *The maximum value of $|L_p(n)|$ for $n < p^k$ occurs at $n = k \cdot \sigma(p^{k-1})$ with value*

$$\max_{n < p^k} |L_p(n)| = k \cdot \max_{n < p} |L_p(n)|,$$

where $\sigma(n)$ is the sum of the divisors of n .

Proof. This follows directly from Lemma 2.30. \square

Corollary 2.32. *If p is an odd prime, then $|L_p(n)| \ll \log n$; furthermore,*

$$\max_{n \leq x} |L_p(x)| \asymp \log x.$$

This last corollary begs the question: *what can be said about the growth of $\left| \sum_{n \leq x} f(n) \right|$ for any function $f \in \mathcal{F}(\{-1, 1\})$?* Presumably this quantity is unbounded for all such f , though this is presently unknown.

Chapter 3

Mahler's method via two examples

Before considering values and power series of more general functions in $\mathcal{F}(\{-1, 1\})$, we present two detailed examples using Mahler's transcendence methods. The proofs here were inspired by Dekking's proof of the transcendence of the Thue–Morse number [34].

The two examples discussed here concern power series, $F(z) \in \mathbb{C}[[z]]$, which satisfy two very different types of functional equations similar to

$$(k \geq 2) \quad F(z^k) = R(z)F(z) \quad \text{and} \quad F(z^k) = F(z) + R(z),$$

where $R(z) \in \mathbb{Z}(z)$.

3.1 Stern's diatomic sequence

The *Stern sequence*, sometimes called Stern's diatomic sequence, $(a(n))_{n \geq 0}$ is given by $a(0) = 0$, $a(1) = 1$, and when $n \geq 1$, by

$$a(2n) = a(n) \quad \text{and} \quad a(2n + 1) = a(n) + a(n + 1).$$

Properties of this sequence have been studied by many authors; for references see [36]. The Stern sequence is A002487 in Sloane's list (see <http://www.research.att.com/~njas/sequences/A000108>). In the article cited above, Dilcher and Stolarsky introduced and studied a polynomial analogue of the Stern sequence, defined by $a(0; x) = 0$, $a(1; x) = 1$, and when $n \geq 1$, by

$$a(2n; x) = a(n; x^2) \quad \text{and} \quad a(2n + 1; x) = x a(n; x^2) + a(n + 1; x^2).$$

We call $a(n; x)$ the n th *Stern polynomial*. Denote by $A(z)$ and $A(x, z)$ the generating functions of the Stern sequence and Stern polynomials, respectively.

In this section, we prove that these generating functions are transcendental.

There are some special subsequences of $(a(n))_{n \geq 0}$ of interest. It is known (see Lehmer [70] and Lind [71]) that the maximum value of $a(m)$ in the interval $2^{n-2} \leq m \leq 2^{n-1}$ is the n th Fibonacci number F_n and that this maximum occurs at

$$m = \frac{1}{3}(2^n - (-1)^n) \quad \text{and} \quad m = \frac{1}{3}(5 \cdot 2^{n-2} + (-1)^n).$$

Dilcher and Stolarsky [35] set

$$\alpha_n := \frac{1}{3}(2^n - (-1)^n) \quad (n \geq 0), \quad \beta_n := \frac{1}{3}(5 \cdot 2^{n-2} + (-1)^n) \quad (n \geq 2),$$

and define for $n \geq 0$

$$f_n(q) := a(\alpha_n; q)$$

and for $n \geq 2$

$$\bar{f}_n(q) := a(\beta_n; q).$$

Throughout the paper [35] the authors study properties of f_n and \bar{f}_n , finding functional equations and other such relationships. They are particularly concerned with the functions F and G defined as follows.

Definition 3.1. For complex q with $|q| < 1$ we define

$$\begin{aligned} F(q) &:= \lim_{n \rightarrow \infty} f_{2n}(q) = \lim_{n \rightarrow \infty} \bar{f}_{2n+1}(q) \\ &= 1 + q + q^2 + q^5 + q^6 + q^8 + q^9 + q^{10} + q^{21} + q^{22} + q^{24} + \dots, \\ G(q) &:= \lim_{n \rightarrow \infty} f_{2n+1}(q) = \lim_{n \rightarrow \infty} \bar{f}_{2n}(q) \\ &= 1 + q + q^3 + q^4 + q^5 + q^{11} + q^{12} + q^{13} + q^{16} + q^{17} + q^{19} + \dots. \end{aligned}$$

In a remark in [35], Dilcher and Stolarsky ask about the transcendence of F and G but make no conclusions. We resolve this question: these functions are transcendental.

3.1.1 Transcendence of $A(z)$

Recall that $A(z) := \sum_{n \geq 0} a(n)z^n$. Using the definition of the Stern sequence, we have

$$\begin{aligned}
 A(z) &= \sum_{n \geq 0} a(2n)z^{2n} + \sum_{n \geq 0} a(2n+1)z^{2n+1} \\
 &= \sum_{n \geq 0} a(n)z^{2n} + \sum_{n \geq 0} a(n)z^{2n+1} + \sum_{n \geq 0} a(n+1)z^{2n+1} \\
 &= A(z^2) + zA(z^2) + \sum_{n \geq 0} a(n)z^{2n-1} \\
 &= A(z^2) \left(1 + z + \frac{1}{z} \right),
 \end{aligned}$$

which gives the following lemma (this result can also be derived from (2.9) in [36]).

Lemma 3.2. *If $A(z)$ is the generating function of the Stern sequence, then*

$$A(z^2) = A(z) \left(\frac{z}{z^2 + z + 1} \right).$$

Theorem 3.3. *The function $A(z)$ is transcendental over $\mathbb{C}(z)$.*

Proof. Towards a contradiction, suppose that $A(z)$ is algebraic and satisfies, say,

$$q_n(z)A(z)^n + q_{n-1}(z)A(z)^{n-1} + \cdots + q_0(z) = 0 \quad (3.1)$$

where $q_i(z) \in \mathbb{C}[z]$, $\gcd(q_n(z), q_{n-1}(z), \dots, q_0(z)) = 1$, and n is chosen minimally. Using the functional equation, we have

$$0 = \sum_{k=0}^n q_k(z^2)A(z^2)^k = \sum_{k=0}^n q_k(z^2)A(z)^k \left(\frac{z}{z^2 + z + 1} \right)^k,$$

and upon multiplying by $(z^2 + z + 1)^n$ we obtain

$$0 = \sum_{k=0}^n q_k(z^2)(z^2 + z + 1)^{n-k} z^k A(z)^k.$$

Thus

$$\begin{aligned}
 0 &= z^n q_n(z^2) \sum_{k=0}^n q_k(z)A(z)^k - q_n(z) \sum_{k=0}^n q_k(z^2)(z^2 + z + 1)^{n-k} z^k A(z)^k \\
 &= \sum_{k=0}^n \left[q_n(z^2)q_k(z)z^n - q_n(z)q_k(z^2)(z^2 + z + 1)^{n-k} z^k \right] A(z)^k.
 \end{aligned} \quad (3.2)$$

The coefficient of $A(z)^n$ in (3.2) is

$$q_n(z^2)q_n(z)z^n - q_n(z)q_n(z^2)z^n = 0,$$

so that

$$0 = \sum_{k=0}^{n-1} \left[q_n(z^2)q_k(z)z^n - q_n(z)q_k(z^2)(z^2 + z + 1)^{n-k}z^k \right] A(z)^k.$$

The minimality of n gives

$$q_n(z^2)q_k(z)z^n = q_n(z)q_k(z^2)(z^2 + z + 1)^{n-k}z^k \quad (3.3)$$

for $k = 0, 1, \dots, n-1$.

Recall that $\gcd(q_n(z), q_{n-1}(z), \dots, q_0(z)) = 1$, so that $\gcd(q_n(z^2), q_{n-1}(z^2), \dots, q_0(z^2)) = 1$. Denote the primitive cube roots of unity by ω and ω^2 . We have $(z-\omega)(z-\omega^2) = z^2+z+1$. Equation (3.3) gives for all $i = 0, 1, \dots, n$ that both

$$z - \omega \mid q_i(z) \iff z - \omega^2 \mid q_i(z^2) \quad (3.4)$$

and

$$z - \omega^2 \mid q_i(z) \iff z - \omega \mid q_i(z^2). \quad (3.5)$$

Denote by $N_a(p(z))$ the multiplicity of the root $z = a$ of $p(z)$. Also note that for all $i = 0, 1, \dots, n$ the equations (3.4) and (3.5) give

$$N_\omega(q_i(z)) = N_{\omega^2}(q_i(z^2)) \quad \text{and} \quad N_\omega(q_i(z^2)) = N_{\omega^2}(q_i(z)).$$

Then (3.3), (3.4) and (3.5) give the system of equations

$$N_\omega(q_n(z^2)) + N_\omega(q_k(z)) = N_\omega(q_n(z)) + N_\omega(q_k(z^2)) + n - k \quad (3.6)$$

$$N_\omega(q_n(z)) + N_\omega(q_k(z^2)) = N_\omega(q_n(z^2)) + N_\omega(q_k(z)) + n - k. \quad (3.7)$$

Substitution of (3.5) into (3.4) gives $n = k$, a contradiction. Therefore, $A(z)$ is transcendental. \square

We proceed to show that the values of $A(z)$ at algebraic $z \in \mathbb{C}$ are transcendental. We use a theorem of Mahler [73], as taken from Nishioka's book [83]. For completeness, a full proof of this theorem is contained in Appendix A. Here \mathbf{I} is the set of algebraic integers over

\mathbb{Q} , K is an algebraic number field, $\mathbf{I}_K = K \cap \mathbf{I}$, and $f(z) \in K[[z]]$ with radius of convergence $R > 0$ satisfying the functional equation for an integer $d > 1$,

$$f(z^d) = \frac{\sum_{i=0}^m b_i(z) f(z)^i}{\sum_{i=0}^m c_i(z) f(z)^i}, \quad m < d, \quad b_i(z), c_i(z) \in \mathbf{I}_K[z],$$

and $\Delta(z) := \text{Res}(B, C)$ is the resultant of $B(u) = \sum_{i=0}^m b_i(z) u^i$ and $C(u) = \sum_{i=0}^m c_i(z) u^i$ as polynomials in u .

Theorem 3.4 (Mahler [73]). *Assume that $f(z)$ is not algebraic over $K(z)$. If α is an algebraic number with $0 < |\alpha| < \min\{1, R\}$ and $\Delta(\alpha^{d^k}) \neq 0$ ($k \geq 0$), then $f(\alpha)$ is transcendental.*

Using Mahler's Theorem we prove

Theorem 3.5. *If $\alpha \neq 0$ is an algebraic number with $0 < |\alpha| < 1$, then $A(\alpha)$ is transcendental over \mathbb{Q} .*

Proof. Let $\alpha \neq 0$ be an algebraic number with $0 < |\alpha| < 1$. Lemma 3.2 gives

$$A(z^2) = \frac{zA(z)}{z^2 + z + 1},$$

so that the resultant (in the variable u) is

$$\Delta(\alpha^{2^k}) = \text{Res}(zu, z^2 + z + 1) \Big|_{z=\alpha^{2^k}} = \alpha^{2^{k+1}} + \alpha^{2^k} + 1,$$

which is non-zero for every $k \geq 0$. The result follows. \square

3.1.2 Transcendence of $A(x, z)$

Theorem 3.6. *The function $A(x, z)$ is transcendental.*

To prove Theorem 3.6 we need the following straightforward lemma concerning values of algebraic functions.

Lemma 3.7. *If f is a power series expansion of an algebraic function, and $\alpha \neq 0$ is an algebraic number within the radius of convergence of f , then $f(\alpha)$ is algebraic.*

Proof. Suppose that f is a power series expansion of an algebraic function of degree n with $\sum_{k=0}^n q_k(z) f(z)^k = 0$, and $\alpha \neq 0$ is an algebraic number within the radius of convergence of f . Since the $q_k(z)$ are polynomials, $q_k(\alpha)$ is an element in the required ring/field and so $f(\alpha)$ satisfies the algebraic equation $\sum_{k=0}^n q_k(\alpha) f(\alpha)^k = 0$. Thus $f(\alpha)$ is algebraic. \square

Proof of Theorem 3.6. From the definitions of the Stern sequence and Stern polynomials, we have $a(n; 1) = a(n)$ so that

$$A(1, z) = A(z).$$

Lemma 3.7 tells us that to prove the transcendence of $A(x, z)$ we need only find algebraic values of x and z so that at this value of $A(x, z)$ is not algebraic.

Consider $x = 1$ and $z = \frac{1}{2}$. Then

$$A\left(1, \frac{1}{2}\right) = A\left(\frac{1}{2}\right),$$

which by Theorem 3.5 is not algebraic, and we have proven the desired result. \square

3.1.3 Transcendence of $F(q)$ and $G(q)$

We now turn to the functions $F(q)$ and $G(q)$ as introduced in Definition 3.1.

Theorem 3.8. *The functions $F(q)$ and $G(q)$ defined above are transcendental over $\mathbb{Q}(z)$.*

To prove this theorem we will need a couple of results from elsewhere. The first is a theorem of Fatou [48] and the second from Dilcher and Stolarsky [35].

Theorem 3.9 (Fatou, 1906). *A power series whose coefficients take only finitely many values that belong to \mathbb{Q} is either rational or transcendental.*

Proposition 3.10 (Dilcher and Stolarsky [35]). *The coefficients of $f_n(q)$ and $\bar{f}_n(q)$ are 0 or 1, and for $k \geq 1$ we have*

$$\deg f_{2k}(q) = \alpha_{2k-1} - 1, \quad \deg f_{2k+1}(q) = \alpha_{2k},$$

$$\deg \bar{f}_{2k}(q) = \beta_{2k-1}, \quad \deg \bar{f}_{2k+1}(q) = \beta_{2k} - 1,$$

where α_n and β_n are as defined previously.

Fatou's theorem is very useful in transcendence proofs. Many different proofs have been given, the first by Fatou in 1906 [48], Allouche in 1999 [3] and again by Borwein and Coons in 2009 [16], whose proof is presented in Chapter 4 of this thesis.

Proof of Theorem 3.8. Recall that $G(q) = \lim_{n \rightarrow \infty} f_{2k+1}(q)$. Proposition 5.3 gives

$$\deg f_{2k+1}(q) = \alpha_{2k} = \frac{1}{3}(2^{2k} - 1).$$

The number of terms of f_{2k+1} is

$$f_{2k+1}(1) = F_{2k+1} = [\varphi^{2k+1}],$$

where the second equality is true for $k \geq 2$, $[x]$ represents the greatest integer less than or equal to x , and φ is the golden ratio. It is important to note that the degree is growing much faster than the number of terms; this property along with Fatou's theorem is enough to give the result.

Consider the degree of f_{2k+1} as k grows. We have

$$\deg f_{2k+3}(q) - \deg f_{2k+1}(q) = \alpha_{2k+2} - \alpha_{2k} = 2^{2k}.$$

If we write the polynomial

$$f_{2k+3}(q) - f_{2k+1}(q) = \varepsilon_{\alpha_{2k+1}} q^{\alpha_{2k+1}} + \varepsilon_{\alpha_{2k+2}} q^{\alpha_{2k+2}} + \cdots + \varepsilon_{\alpha_{2k+2}} q^{\alpha_{2k+2}},$$

then the proportion of ε_i s that are 1 is

$$\frac{[\varphi^{2k+2}]}{2^{2k}}$$

which approaches 0 as k goes to infinity.

Let h be a given positive integer. Then there exists a $k := k(h)$ for which

$$\frac{[\varphi^{2k+2}]}{2^{2k}} < \frac{1}{h^2}.$$

Thus, for this k , the polynomial $f_{2k+3}(q) - f_{2k+1}(q)$ contains at least h consecutive ε_i s with $\varepsilon_i = 0$.

Now pick $q = \frac{1}{2}$. Then $G(1/2)$ is a binary number with (in binary notation)

$$G(1/2) = \varepsilon_0.\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\dots = 1.1100110111\dots$$

The previous paragraph tells us that there are arbitrarily long runs of zeros in the binary expansion of $G(1/2)$. Since $\deg f_{2k+1}(q) \rightarrow \infty$ with k , there are infinitely many ones in the binary expansion of $G(1/2)$ and so $G(1/2)$ is an irrational number. Hence $G(q)$ is not a rational function. Application of Fatou's theorem gives the desired result. \square

3.2 Golomb's series

Golomb proved in [49] that the values of the functions

$$\sum_{n \geq 0} \frac{z^{2^n}}{1 + z^{2^n}} \quad \text{and} \quad \sum_{n \geq 0} \frac{z^{2^n}}{1 - z^{2^n}}$$

are irrational at $z = \frac{1}{t}$ for $t = 2, 3, 4, \dots$, the interesting special case of which is that the sum of the reciprocals of the Fermat numbers is irrational. Schwarz [88] has given results on series of the form

$$G_k(z) := \sum_{n \geq 0} \frac{z^{k^n}}{1 - z^{k^n}}.$$

In particular, he proved that if k, t and b are integers satisfying

$$k \geq 2, \quad t \geq 2, \quad \text{and} \quad 1 \leq b < t^{1-1/k},$$

then the number

$$G_k(bt^{-1}) = \sum_{n \geq 0} \frac{b^{k^n}}{t^{k^n} - b^{k^n}}$$

is irrational. Schwarz also showed that for $k, t, b \in \mathbb{N}$ with $k > 2$, $t \geq 2$ and $1 \leq b < t^{1-5/2k}$ the number $G_k(bt^{-1})$ is transcendental. The case $k = 2$ proved to be more difficult, though he was able to show that for an integer $t \geq 2$ the number $G_2(t^{-1})$ is not algebraic of the second degree.

Schwarz also remarks in [88] that, “the irrationality of

$$\sum_{n \geq 0} b^{k^n} (t^{k^n} + b^{k^n})^{-1}$$

for $k > 2$ is unsettled.” In our notation, this is $F_k(bt^{-1})$.

Recently, Duverney [38] has proven the transcendence of $G_2(t^{-1})$ for integers $t \geq 2$ and extended Schwarz's transcendence results for the case $k = 2$. He proved the following theorem.

Theorem 3.11 (Duverney [38]). *Let $a \geq 2$ be an integer and let b_n be a sequence of integers satisfying $|b_n| = O(\eta^{-2^n})$ for every $\eta \in (0, 1)$. Suppose that $a^{2^n} + b_n \neq 0$ for every $n \in \mathbb{N}$. Then the number*

$$S = \sum_{n \geq 0} \frac{1}{a^{2^n} + b_n}$$

is transcendental.

We extend Schwarz's results further (to the best possible); in particular, we prove that for all $k \geq 2$ the series $G_k(z) = \sum_{n \geq 0} z^{k^n} (1 - z^{k^n})^{-1}$ is transcendental for all algebraic numbers z with $|z| < 1$. We also prove the same result for $F_k(z) = \sum_{n \geq 0} z^{k^n} (1 + z^{k^n})^{-1}$ which settles the irrationality question of Schwarz's remark. These results were known to Mahler (see [73, 74, 75, 76]), though our proofs of the function's transcendence are new and elementary, coming from the proof of our main result; no linear algebra or differential calculus is used.

The main result of this section is that a non-zero power series $F(z) \in \mathbb{C}[[z]]$ satisfying

$$F(z^d) = F(z) + \frac{A(z)}{B(z)},$$

where $A(z), B(z) \in \mathbb{C}[z]$ with $A(z) \neq 0$ and $\deg A(z), \deg B(z) < d$ is transcendental over $\mathbb{C}(z)$. This extends a theorem of Nishioka [82] that states that $F(z)$ is either transcendental or rational.

3.2.1 A general transcendence theorem

Nishioka [82] has shown

Theorem 3.12 (Nishioka [82]). *Suppose that $F(z) \in \mathbb{C}[[z]]$ satisfies one of the following for an integer $d > 1$.*

(i) $F(z^d) = \varphi(z, F(z)),$

(ii) $F(z) = \varphi(z, F(z^d)),$

where $\varphi(z, u)$ is a rational function in z, u over \mathbb{C} . If $F(z)$ is algebraic over $\mathbb{C}(z)$, then $F(z) \in \mathbb{C}(z)$.

Nishioka's proof of Theorem 3.12 relies on deep ideas from algebraic number theory. In this section we provide an elementary proof of a special case of Theorem 3.12. In this special case, we are able to refine the conclusion by eliminating the possibility of $F(z)$ being a rational function.

Theorem 3.13. *If $F(z)$ is a power series in $\mathbb{C}[[z]]$ satisfying*

$$F(z^d) = F(z) + \frac{A(z)}{B(z)},$$

where $d \geq 2$, $A(z), B(z) \in \mathbb{C}[z]$ with $A(z) \neq 0$ and $\deg A(z), \deg B(z) < d$, then $F(z)$ is transcendental over $\mathbb{C}(z)$.

Proof. Suppose that the power series $F(z)$ is algebraic, and satisfies

$$\sum_{r=0}^n q_r(z) F(z)^r \equiv 0 \quad (3.8)$$

where the $q_r(z)$ are rational functions with $q_n(z) = 1$ and $n \geq 1$ is chosen minimally. The notation \equiv is used to mean identically equal.

Substituting z^d into (3.8) and using the functional relation gives

$$0 \equiv \sum_{r=0}^n q_r(z^d) F(z^d)^r = \sum_{r=0}^n q_r(z^d) \left(F(z) + \frac{A(z)}{B(z)} \right)^r.$$

Without loss of generality, suppose $B(z)$ is monic. Multiplying by $B(z)^n$ to clear fractions as well as an application of the binomial theorem yields

$$\begin{aligned} 0 &\equiv \sum_{r=0}^n q_r(z^d) B(z)^{n-r} (B(z)F(z) + A(z))^r \\ &= \sum_{r=0}^n q_r(z^d) B(z)^{n-r} \sum_{j=0}^r \binom{r}{j} B(z)^j F(z)^j A(z)^{r-j}. \end{aligned} \quad (3.9)$$

Taking the difference between (3.9) and $B(z)^n$ times (3.8) gives

$$Q(z) := \sum_{r=0}^n q_r(z^d) B(z)^{n-r} \sum_{j=0}^r \binom{r}{j} B(z)^j F(z)^j A(z)^{r-j} - B(z)^n \sum_{r=0}^n q_r(z) F(z)^r \equiv 0. \quad (3.10)$$

Note that we may also write

$$Q(z) = \sum_{m=0}^n h_m(z) F(z)^m \equiv 0,$$

for some polynomials $h_0(z), \dots, h_n(z) \in \mathbb{C}[z]$.

We determine $h_n(z)$. The only term in $Q(z)$ that can contribute to the coefficient of $F(z)^n$ is the term with $r = n$ of the sum (3.10), which recalling that $q_n(z) = 1$ is

$$\sum_{j=0}^n \binom{n}{j} B(z)^j F(z)^j A(z)^{n-j} - B(z)^n F(z)^n,$$

and only the $j = n$ term here contributes. Hence

$$h_n(z) = \binom{n}{n} B(z)^n A(z)^{n-n} - B(z)^n = 0,$$

so that $Q(z) = \sum_{m=0}^{n-1} h_m(z)F(z)^m \equiv 0$. Since n was chosen minimally, $h_m(z) \equiv 0$ for all $m = 0, 1, \dots, n-1$.

Using (3.10) we have that

$$h_m(z) = \sum_{r=m}^n \binom{r}{m} q_r(z^d) B(z)^{n-r+m} A(z)^{r-m} - B(z)^n q_m(z).$$

Since $h_{n-1}(z) \equiv 0$ this gives

$$\sum_{r=n-1}^n \binom{r}{n-1} q_r(z^d) B(z)^{n-r+(n-1)} A(z)^{r-(n-1)} = B(z)^n q_{n-1}(z)$$

so that with the removal of shared factors, recalling $q_n(z) = 1$, we have the identity

$$q_{n-1}(z^d)B(z) + nA(z) = B(z)q_{n-1}(z). \quad (3.11)$$

Write $q_{n-1}(z) = \frac{\alpha(z)}{\beta(z)}$ where $\alpha(z), \beta(z) \in \mathbb{C}[z]$ with $\gcd(\alpha(z), \beta(z)) = 1$ and $\beta(z)$ monic. Denote $g(z) := \gcd(\beta(z), \beta(z^d))$, so that $\frac{\beta(z)}{g(z)}, \frac{\beta(z^d)}{g(z)} \in \mathbb{C}[z]$ and hence $\gcd\left(\frac{\beta(z)}{g(z)}, \frac{\beta(z^d)}{g(z)}\right) = 1$. Then (3.11) becomes

$$\left(\frac{\beta(z)}{g(z)}\right) \alpha(z^d)B(z) + n \left(\frac{\beta(z)}{g(z)}\right) \beta(z^d)A(z) = \left(\frac{\beta(z^d)}{g(z)}\right) B(z)\alpha(z). \quad (3.12)$$

Thus $\frac{\beta(z)}{g(z)} | B(z)$. Also, since $\frac{\beta(z)}{g(z)}\beta(z^d) = \beta(z)\frac{\beta(z^d)}{g(z)}$ we have $\frac{\beta(z^d)}{g(z)} | B(z)$.

Equation (3.11) yields $\beta(z^d) | \beta(z)\alpha(z^d)B(z)$, which implies that

$$d \cdot \deg \beta(z) \leq \deg \beta(z) + \deg B(z) < \deg \beta(z) + d.$$

Hence

$$0 \leq \deg \beta(z) < 1 + \frac{1}{d-1}.$$

Since $d \geq 2$, either $\deg \beta(z) = 0$ or $\deg \beta(z) = 1$.

Suppose $\deg \beta(z) = 0$. Hence $\beta(z) = \beta(z^d) \in \mathbb{C}$; write $\beta := \beta(z)$. Now (3.12) becomes

$$\alpha(z^d)B(z) + n\beta A(z) = B(z)\alpha(z). \quad (3.13)$$

Thus $B(z) | n\beta$ which gives $\deg B(z) = 0$. Write $B := B(z)$. Thus (3.13) becomes

$$\alpha(z^d)B + n\beta A(z) = B\alpha(z), \quad (3.14)$$

which implies that $d \cdot \deg \alpha(z) = \deg A(z) < d$ so that $\deg \alpha(z) = 0$. Eq. (3.14) and $\deg \alpha(z) = 0$ imply that $A(z) = 0$ which is impossible.

Now suppose $\deg \beta(z) = 1$ and write $\beta(z) = z - \beta$. Comparing degrees in (3.12) implies that $\deg \alpha(z) \leq 1$.

Recall that both $\frac{\beta(z)}{g(z)} | B(z)$ and $\frac{\beta(z^d)}{g(z)} | B(z)$. Since $\gcd\left(\frac{\beta(z)}{g(z)}, \frac{\beta(z^d)}{g(z)}\right) = 1$, we have that $\left(\frac{\beta(z)}{g(z)} \cdot \frac{\beta(z^d)}{g(z)}\right) | B(z)$. The bound $\deg B(z) < d$ and $\deg g(z) = 1$ give $g(z) = \beta(z)$. Since $\frac{\beta(z^d)}{g(z)} = \frac{\beta(z^d)}{\beta(z)} | B(z)$ and both $\beta(z)$ and $B(z)$ are monic with $\deg B(z) < d$, we have

$$\frac{\beta(z^d)}{\beta(z)} = B(z).$$

Suppose that $\deg \alpha(z) = 1$. Write $\alpha(z) = \delta(z - \alpha)$ and note that $\beta \neq \alpha$. In this case, solving (3.12) for $A(z)$ gives

$$A(z) = \frac{\delta(\beta - \alpha)z(z^{d-1} - 1)}{n(z - \beta)^2} \in \mathbb{C}[z].$$

Since $A(z) \in \mathbb{C}[z]$ we have that $(z - \beta)^2 | (z^{d-1} - 1)$, which we may rewrite as

$$(z - \beta)^2 \left| \prod_{k=0}^{d-2} \left(z - e^{2\pi i \frac{k}{d-1}}\right)\right.$$

This is not possible since $e^{2\pi i \frac{l}{d-1}} \neq e^{2\pi i \frac{m}{d-1}}$ for any l, m with $0 \leq l < m \leq d - 2$ (that is, $(d - 1)$ -th roots of unity are distinct); hence $\deg \alpha(z) = 0$.

If $\deg \alpha(z) = 0$, write $\alpha := \alpha(z)$. Then writing $\beta(z) = z - \beta$ and solving (3.12) for $A(z)$ we have that

$$A(z) = \frac{\alpha z(z^{d-1} - 1)}{n(z - \beta)^2} \notin \mathbb{C}[z],$$

which contradicts that $A(z) \in \mathbb{C}[z]$. □

Corollary 3.14. *There is no rational function $F(z)$ in $\mathbb{C}(z)$ satisfying*

$$F(z^d) = F(z) + \frac{A(z)}{B(z)},$$

where $d \geq 2$, $A(z), B(z) \in \mathbb{C}[z]$ with $A(z) \neq 0$ and $\deg A(z), \deg B(z) < d$.

3.2.2 The series $G_k(z)$ and $F_k(z)$

To prove the transcendence results surrounding $G_k(z)$ and $F_k(z)$, we apply Theorem 3.13 as well as Theorem 3.4.

Consider the functional equation $f(z^k) = f(z) - \frac{z}{1-z}$ with $k \geq 2$. Repeated application yields

$$f(z^{k^m}) = f(z^{k^{m-1}}) - \frac{z^{k^{m-1}}}{1-z^{k^m}} = f(z) - \sum_{n=1}^m \frac{z^{k^{m-n}}}{1-z^{k^{m-n}}}.$$

Changing the index and setting $W_m(z) := \sum_{n=0}^{m-1} \frac{z^{k^n}}{1-z^{k^n}}$ gives

$$f(z) = f(z^{k^m}) + W_m(z).$$

In the region $|z| < 1$, we have

$$f(z) = \lim_{m \rightarrow \infty} [f(z^{k^m}) + W_m(z)] = \sum_{n \geq 0} \frac{z^{k^n}}{1-z^{k^n}} = G_k(z).$$

This proves the following lemma.

Lemma 3.15. *For $|z| < 1$, the function $G_k(z)$ satisfies the functional equation*

$$G_k(z^k) = G_k(z) + \frac{z}{z-1}.$$

As a corollary of Theorem 3.13, we have

Corollary 3.16. *For $k \geq 2$, the function $G_k(z)$ is transcendental over $\mathbb{C}(z)$.*

To get the transcendence of the associated numbers, we use Mahler's Theorem.

Proposition 3.17. *If $k \geq 2$ and α is algebraic with $0 < |\alpha| < 1$, then $G_k(\alpha)$ is transcendental over \mathbb{Q} .*

Proof. Lemma 3.15 gives the functional equation

$$G_k(z^k) = \frac{(1-z)G_k(z) - z}{1-z},$$

so that, in the language of Theorem 3.4, we have

$$A(u) = (1-z)u - z \quad \text{and} \quad B(u) = 1 - z,$$

$m = 1 < k = d$, and $a_i(z), b_i(z) \in \mathbf{I}_K[z]$. Since $B(u)$ is a constant polynomial in u ,

$$\Delta(z) := \text{Res}(A, B) = 1 - z.$$

Let $|\alpha| < 1$ be algebraic; it is immediate that

$$\Delta(\alpha^{k^n}) = 1 - \alpha^{k^n} \neq 0 \quad (n \geq 0).$$

Since $G_k(z)$ is not algebraic over $\mathbb{C}(z)$ (as supplied by Corollary 3.16), applying Theorem 3.4, we have that $G_k(\alpha)$ is transcendental over \mathbb{Q} . \square

Corollary 3.18. *If $k, b, t \in \mathbb{N}$ with $k \geq 2$ and $0 < b < t$, then the number $G_k(bt^{-1})$ is transcendental over \mathbb{Q} .*

Proof. Set $\alpha = b/t$ in Theorem 3.17. \square

We turn now to the series

$$F_k(z) = \sum_{n \geq 0} \frac{z^{k^n}}{1 + z^{k^n}}.$$

Similar to $G_k(z)$, the function $F_k(z)$ satisfies a functional equation,

$$F_k(z^k) = F_k(z) - \frac{z}{z+1}.$$

Theorem 3.13 yields

Corollary 3.19. *The function $F_k(z)$ is transcendental over $\mathbb{C}(z)$.*

As before, Mahler's Theorem gives

Proposition 3.20. *For $k \geq 2$ and $z = \alpha$ an algebraic number with $0 < |\alpha| < 1$, $F_k(\alpha)$ is transcendental over \mathbb{Q} .*

Corollary 3.21. *If $k, b, t \in \mathbb{N}$ with $k \geq 2$ and $1 \leq b < t$, then the number $F_k(bt^{-1})$ is transcendental over \mathbb{Q} .*

Remark 3.22. For some more recent work concerning results like Nishioka's Theorem 3.12, for more general algebraic number fields, see [40]; this paper also contains a number of current references to results in this area. Also, concerning functions similar to $G_k(z)$ and $F_k(z)$ above, Duverney, Kanoko, and Tanaka [39] have given a complete classification of those series

$$f(z) := \sum_{k \geq 0} \frac{a^k z^{d^k}}{1 + bz^{d^k} + cz^{2d^k}} \in C[[z]]$$

that are transcendental over $C(z)$ where C is a field of characteristic 0, $d \geq 2$, and $a, b, c \in C$ with $a \neq 0$.

Chapter 4

Irrationality and transcendence

In this chapter we return to our investigation of functions $f \in \mathcal{F}(\{-1, 1\})$ by considering formal power series with coefficients $f(n)$ as well as the values of power series whose coefficients come from a certain subset of $\mathcal{F}(\{-1, 1\})$. The results of this chapter are taken from two joint publications ([15] and [16]) with Peter Borwein.

4.1 Formal power series

In 1945, Duffin and Schaeffer [37] proved that

Theorem 4.1 (Duffin and Schaeffer [37]). *A power series that is bounded in a sector and has coefficients from a finite subset is a rational function.*

Their proof is relatively indirect. In [17], Borwein, Erdélyi, and Littman gave a shorter direct proof.

The theorem of Duffin and Schaeffer is a generalization of a result of Szegő who proved in 1922 that *a power series whose coefficients assume only finitely many values and which can be extended analytically beyond the unit circle is a rational function.*

In 1906, Fatou [48] proved, and in 1999, Allouche [3] reproved using a deep result of Cobham [25], that

Theorem 4.2 (Fatou [48]). *A power series whose coefficients take only finitely many rational values is either rational or transcendental.*

In this section, we give a new proof of Fatou's theorem and apply it to show that if $f \in \mathcal{F}(\{-1, 1\})$ and f is not identically 1, then $\sum_{n \geq 1} f(n)z^n$ is transcendental over $\mathbb{Z}(z)$.

As a specific example, we show the transcendence of the series $\sum_{n \geq 1} \lambda(n)z^n$. We also use Fatou's theorem to show that $\sum_{n \geq 1} \mu(n)z^n$. Here λ and μ are the Liouville and Möbius functions, respectively.

We will need the following quantitative version of the Fundamental Theorem of Algebra [18, Theorem 1.2.1].

Lemma 4.3. *The polynomial*

$$p(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \in \mathbb{C}[z], \quad a_n \neq 0$$

has exactly n zeros in \mathbb{C} . These all lie in the open disk of radius r centered at the origin, where

$$r := 1 + \max_{0 \leq k \leq n-1} \frac{|a_k|}{|a_n|}.$$

Proof of Theorem 4.2. Suppose that $f(z)$ is a power series with coefficients that take only finitely many rational values, and satisfies

$$a_n(z)f(z)^n + a_{n-1}(z)f(z)^{n-1} + \cdots + a_0(z) = 0$$

where each $a_i(z)$ is a polynomial with integer coefficients. Since the leading coefficient $a_n(z)$ of this polynomial equation is a polynomial, it has finitely many zeros. Hence there is a sector S of the open unit disk where $|a_n(z)|$ is bounded away from zero uniformly. The modulus of each other coefficient $a_k(z)$ is clearly uniformly bounded above on S . Now apply Lemma 4.3 to conclude that $|f(z)|$ is bounded on S , so the result of Duffin and Schaeffer applies. \square

As before, denote by μ the Möbius function, and by λ the Liouville function. Recall that λ is the unique completely multiplicative function defined by $\lambda(p) = -1$ for all primes p .

In [8], it is shown that the formal power series $\sum_{n \geq 1} \lambda(n)z^n, \sum_{n \geq 1} \mu(n)z^n \in \mathbb{Z}[[z]]$ are irrational over $\mathbb{Z}(z)$ (along with various other multiplicative functions). We proceed by proving that these two power series are transcendental over $\mathbb{Z}(z)$. The transcendence of $\sum_{n \geq 1} \lambda(n)z^n$ is stated as a corollary to the following general theorem.

Theorem 4.4. *Let $f : \mathbb{N} \rightarrow \{-1, 1\}$ be a completely multiplicative function with the property that for some prime p , $f(p) = -1$. Then $\sum_{n \geq 1} f(n)z^n \in \mathbb{Z}[[z]]$ is transcendental over $\mathbb{Z}(z)$.*

Proof. In light of Theorem 4.2, we need only demonstrate that for a completely multiplicative function $f : \mathbb{N} \rightarrow \{-1, 1\}$ such that there is a prime p for which $f(p) = -1$, the sequence of values of f is not eventually periodic. This would show that $\sum_{n \geq 1} f(n)2^{-n}$ is irrational. Denote the sequence of values of f by \mathfrak{F} .

Towards a contradiction, suppose that \mathfrak{F} is eventually periodic, say the sequence is periodic after the M -th term and has period k . Now there is an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $nk > M$. Let p be a prime for which $f(p) = -1$. Then

$$f(pnk) = f(p)f(nk) = -f(nk).$$

But $pnk \equiv nk \pmod{k}$, a contradiction to the eventual k -periodicity of \mathfrak{F} . \square

Corollary 4.5. *If λ is the Liouville function, then the series $\sum_{n \geq 1} \lambda(n)z^n \in \mathbb{Z}[[z]]$ is transcendental over $\mathbb{Z}(z)$.*

Note that Theorem 4.4 does not apply directly to the Möbius function because μ is not completely multiplicative. Recall from the definition that if $p^2 \mid n$ for any prime p , then $\mu(n) = 0$. From this fact alone one may use the Chinese Remainder Theorem to show that sequence of values of the Möbius function contains arbitrarily long runs of zeroes. This in turn gives the irrationality of $\sum_{n \geq 1} \mu(n)z^n$ at $z = \frac{1}{3}$, and hence the following corollary to Theorem 4.2.

Corollary 4.6. *If μ is the Möbius function, then the series $\sum_{n \geq 1} \mu(n)z^n \in \mathbb{Z}[[z]]$ is transcendental over $\mathbb{Z}(z)$.*

The transcendence of the series $\sum_{n \geq 1} f(n)z^n$ over $\mathbb{Z}(z)$ for f equal to each of the multiplicative functions τ_k, σ_k , and φ was shown by Yazdani [98]; the case $f = \mu$ was previously treated by Allouche [3] using a deep result of Cobham [26].

While transcendence results on power series are readily available, there are many open questions concerning their special values. Erdős [43] was interested in the transcendence and irrationality of both $\sum_{n \geq 1} \tau(n)2^{-n}$ and $\sum_{n \geq 1} \varphi(n)2^{-n}$. The irrationality of the first sum was shown by Borwein [11]; the transcendence is not yet known. Questions regarding the algebraic character of $\sum_{n \geq 1} \varphi(n)2^{-n}$ remain open. Nesterenko [79] has shown that the number

$$\sum_{n \geq 1} \frac{\lambda(n)}{2^n - 1}$$

is transcendental, but the transcendence of its close relative, $\sum_{n \geq 1} \lambda(n)2^{-n}$ remains elusive. Using Mahler's method, as described in Chapter 3, in the next section we approach the transcendence of values of power series whose coefficients are given by certain functions in $\mathcal{F}(\{-1, 1\})$.

4.2 Values of power series

Denote the sequence of λ values by \mathfrak{L} and recall that *the Liouville number* l is the binary number

$$l := \sum_{n \geq 1} \left(\frac{1 + \lambda(n)}{2} \right) \frac{1}{2^n} = 0.100101001100011100001 \dots$$

Properties of the number l are properties of the sequence \mathfrak{L} . One noteworthy property is that l is irrational.

Theorem 4.7. *The Liouville number is not rational.*

Proof. See the proof of Theorem 4.4. □

The irrationality of l tells us that the sequence \mathfrak{L} is not eventually periodic. A fundamental question arises, which at present we are unable to answer. We believe that l is transcendental, though this seems currently unapproachable. For other completely multiplicative functions like λ , we can decide transcendence.

In this section, we consider a subset of $\mathcal{F}(\{-1, 1\})$.

As an example, let g be the completely multiplicative function defined on primes p by

$$g_p = \begin{cases} -1 & \text{if } p \equiv 3 \pmod{4} \\ 1 & \text{otherwise.} \end{cases}$$

As the function g takes the value -1 on those primes which are rational Gaussian primes, we call g *the Gaussian Liouville function*. Denote by \mathfrak{G} the sequence of values of g , and define

$$\gamma := \sum_{n \geq 1} \left(\frac{1 + g_n}{2} \right) \frac{1}{2^n}$$

as the *Gaussian Liouville number*. One can show that $g_n = (-1/n)$ where (\cdot/n) is the Jacobi symbol modulo n . The Gaussian Liouville number is easily seen to be irrational. Indeed, it is a corollary to the following generalization of Theorem 4.7.

Theorem 4.8. *Suppose that $f : \mathbb{N} \rightarrow \{-1, 1\}$ is a completely multiplicative function, with $f(p) = -1$ for at least one prime p , and \mathfrak{F} its sequence of values. If $\varphi := \sum_{n \geq 1} \left(\frac{1+f(n)}{2}\right) \frac{1}{2^n}$, then $\varphi \notin \mathbb{Q}$.*

Proof. See the proof of Theorem 4.4. □

Though a proof of transcendence of the Liouville number seems unattainable, it is possible to establish the transcendence of the Gaussian Liouville number and many of its relatives. The proof of this result is contained in Subsection 4.2.2, and rests on the fact that the generating function of the sequence \mathfrak{G} satisfies a useful functional equation (see Lemma 4.12). In addition to providing a nice connection to the theory of finite automata, this functional equation leads to a striking power series representation of the functional equation. It is of interest (an example of such a series representation is given in Subsection 4.2.1), and may lead to a quick transcendence result.

As an example of the usefulness of such a representation, we prove in Subsection 4.2.2 that the generating function $G(z)$ of the Gaussian Liouville sequence is

$$G(z) = \sum_{k \geq 0} \frac{z^{2^k}}{1 + z^{2^{k+1}}}.$$

An ingredient to the proof of the transcendence of the Gaussian Liouville number is the transcendence of the generating function of the sequence \mathfrak{G} . This is easily accomplished using Theorem 4.8 and Fatou's theorem.

The method used in our proof can be used to prove more general results regarding other completely multiplicative functions. Subsection 4.2.3 contains these results. For an account of the properties of these functions see [13].

A few historical remarks are in order. The irrationality of the values of power series similar to those of our investigation have been studied by, among others, Erdős, Golomb, Mahler, and Schwarz. Erdős [43] proved that the series

$$\sum_{n \geq 1} \frac{z^n}{1 - z^n} = \sum_{n \geq 1} d(n)z^n,$$

where $d(n)$ is the divisor counting function, gives irrational values at $z = \frac{1}{t}$ for $t = 2, 3, 4, \dots$, and Allouche [3] has shown this function to be transcendental, but the transcendence of any value is still open. For example, $z = \frac{1}{2}$ presumably gives a transcendental value. Indeed Erdős writes [44],

“It is very annoying that I cannot prove that $\sum_{n \geq 1} \frac{1}{2^n - 3}$ and $\sum_{n \geq 2} \frac{1}{n! - 1}$ are both irrational (one of course expects that $\sum_{n \geq 1} \frac{1}{2^n + t}$ and $\sum_{n \geq 2} \frac{1}{n! + t}$ are irrational and in fact transcendental for every integer t .)”

Partially answering Erdős’ question, Borwein [12] has shown that for $q \in \mathbb{Z}$ with $|q| > 1$ and $c \in \mathbb{Q}$ with $c \neq 0$ and $c \neq -q^n$ ($n \in \mathbb{N}$),

$$\sum_{n \geq 1} \frac{1}{q^n + c} \quad \text{and} \quad \sum_{n \geq 1} \frac{(-1)^n}{q^n + c}$$

are irrational; the special values $c = -1$ and $q = 2$ give that the sum of the reciprocals of the Mersenne numbers is irrational. Later, Golomb proved in [49] that the values of the functions

$$\sum_{n \geq 0} \frac{z^{2^n}}{1 + z^{2^n}} \quad \text{and} \quad \sum_{n \geq 0} \frac{z^{2^n}}{1 - z^{2^n}}$$

are irrational at $z = \frac{1}{t}$ for $t = 2, 3, 4, \dots$. As a special case we obtain that the sum of the inverses of the Fermat numbers is irrational. Transcendence of the sum of the inverses of the Fermat numbers is implied by Duverney’s theorem (Theorem 3.11). Schwarz [88] has given results on series of the form

$$\sum_{n \geq 0} \frac{z^{k^n}}{1 - z^{k^n}}.$$

In particular, he showed that this function is transcendental at certain rational values of z when $k \geq 2$ is an integer. These series are discussed in more detail in Section 3.2 where complete transcendence results are given. We take these results further and prove that for $f \in \mathcal{F}(\{-1, 1\})$ satisfying the recursive relations $f_p = \pm 1$ and $f_{pk+i} = f_i$ for $i = 1, 2, \dots, p - 1$, the series

$$\sum_{k \geq 0} f(n)z^n = \sum_{k \geq 0} \frac{f_p^k \Phi(z)}{1 - z^{p^{k+1}}},$$

where $\Phi(z) = \sum_{i=1}^{p-1} f_i z^{p^k i}$, is transcendental provided f is not identically 1 (see Theorem 4.18 and Proposition 4.19). It is interesting to note that when $f_i = (i/p)$ (for $p \nmid i$) is the Legendre symbol, the polynomial $\Phi(z)$ is the p th degree Fekete polynomial. Patterns in the sequence of values of such f have been studied by Hudson [58, 59].

Mahler’s results are too numerous to mention, and it seems likely that at least some of the historical results mentioned here were known to him as early as the 1920s; see [76]. Mahler was one of the first to consider the links between functional equations and transcendence (see [83]).

4.2.1 The Liouville function for primes 2 modulo 3

As an example of a power series representation of a generating function consider the completely multiplicative function t_n where

$$t_3 = 1 \quad \text{and} \quad t_p = \begin{cases} -1 & \text{if } p \equiv 2 \pmod{3} \\ 1 & \text{if } p \equiv 1 \pmod{3}. \end{cases}$$

We have the relations

$$t_{3n} = t_n, \quad t_{3n+1} = 1, \quad \text{and} \quad t_{3n+2} = -1.$$

Denote the generating function of $(t_n)_{n \in \mathbb{N}}$ as $T(z) = \sum_{n \geq 1} t_n z^n$. Then

$$T(z) = \sum_{n \geq 1} t_{3n} z^{3n} + \sum_{n \geq 0} t_{3n+1} z^{3n+1} + \sum_{n \geq 0} t_{3n+2} z^{3n+2} = T(z^3) + (z - z^2) \frac{1}{1 - z^3},$$

which gives the following result.

Lemma 4.9. *If $T(z) = \sum_{n \geq 1} t_n z^n$, then*

$$T(z^3) = T(z) - \frac{z}{1 + z + z^2}.$$

Using this lemma we have

$$T(z^{3^m}) = T(z) - \sum_{k=0}^{m-1} \frac{z^{3^k}}{1 + z^{3^k} + z^{3^{k+1}}}.$$

Denote the sum by $U_m(z)$; that is,

$$U_m(z) = \sum_{k=0}^{m-1} \frac{z^{3^k}}{1 + z^{3^k} + z^{3^{k+1}}}.$$

If $|z| < 1$, taking the limit as $m \rightarrow \infty$, gives the desired series expression.

Proposition 4.10. *If $|z| < 1$, then the generating function of $(t_n)_{n \in \mathbb{N}}$ has the closed form*

$$T(z) = \sum_{k \geq 0} \frac{z^{3^k}}{1 + z^{3^k} + z^{3^{k+1}}}.$$

Application of the general results proved in Subsection 4.2.3 gives the following result.

Theorem 4.11. *The function $T(z)$ is transcendental over $\mathbb{C}(z)$; furthermore, $T(\alpha)$ is transcendental over \mathbb{Q} for all non-zero algebraic numbers α with $|\alpha| < 1$.*

4.2.2 The Gaussian Liouville function

As before, the *Gaussian Liouville function* g is the completely multiplicative function defined on the primes by

$$g_p = \begin{cases} -1 & \text{if } p \equiv 3 \pmod{4} \\ 1 & \text{otherwise.} \end{cases}$$

Also, denote by \mathfrak{G} the sequence of values of g , and by

$$\gamma := \sum_{n \geq 1} \left(\frac{1 + g_n}{2} \right) \frac{1}{2^n},$$

the *Gaussian Liouville number*.

The first few values of g are

$$\mathfrak{G} = (1, 1, -1, 1, 1, -1, -1, 1, 1, 1, -1, -1, 1, -1, -1, 1, 1, 1, -1, 1, 1, -1, \dots).$$

Elementary observations tell us that the occurrence of primes that are 3 modulo 4 in prime factorizations are fairly predictable. One has the following implications:

$$\begin{aligned} n \equiv 1 \pmod{4} &\Rightarrow g_n = 1 \\ n \equiv 3 \pmod{4} &\Rightarrow g_n = -1 \\ n \equiv 0 \pmod{2} &\Rightarrow g_{2n} = g_n, \end{aligned}$$

which give the recurrence relations for the sequence \mathfrak{G} as

$$g_1 = 1, \quad g_{2n} = g_n, \quad g_{4n+1} = -g_{4n+3}. \quad (4.1)$$

This is not so surprising recalling that $g_n = (-1/n)$ where (\cdot/n) is the Jacobi symbol modulo n .

Let $G(z) = \sum_{n \geq 1} g_n z^n$ be the generating function for the sequence \mathfrak{G} . Note that $G(z)$ is holomorphic inside the unit disk. The recurrence relations in (4.1) lead to a functional equation for $G(z)$.

Lemma 4.12. *If $G(z) = \sum_{n \geq 1} g_n z^n$, then*

$$G(z^2) = G(z) - \frac{z}{1+z^2}.$$

Proof. This is directly given by the recurrences for \mathfrak{G} . We calculate that

$$\begin{aligned}
 G(z) &= \sum_{n \geq 1} g_n z^n = \sum_{n \geq 1} g_{2n} z^{2n} + \sum_{n \geq 1} g_{4n+1} z^{4n+1} + \sum_{n \geq 1} g_{4n+3} z^{4n+3} \\
 &= \sum_{n \geq 1} g_n z^{2n} + z \sum_{n \geq 1} g_{4n+1} z^{4n} + z^3 \sum_{n \geq 1} g_{4n+3} z^{4n} \\
 &= \sum_{n \geq 1} g_n z^{2n} + z \sum_{n \geq 1} g_{4n+1} z^{4n} - z^3 \sum_{n \geq 1} g_{4n+1} z^{4n} \\
 &= G(z^2) + (z - z^3) \sum_{n \geq 1} z^{4n} \\
 &= G(z^2) + \frac{z - z^3}{1 - z^4}.
 \end{aligned}$$

A little arithmetic and rearrangement gives the desired result. \square

Remark 4.13. A functional equation like the one in the above lemma offers a deep interplay with the theory of finite automata. Indeed, if we consider $G(z) \in \mathbb{F}_2[[z]]$, then since $G(z^2) = G(z)^2$, this function is algebraic over $\mathbb{F}_2(z)$; more specifically,

$$G(z)^2 - G(z) + \frac{z}{1 + z^2} = 0.$$

Since this function is algebraic over $\mathbb{F}_2(z)$, by a classical theorem of Christol [22], the sequence $(g(n))_{n \geq 1}$ can be produced by a 2-automaton. For ease of explanation, if we read the base 2 expansion of n backwards and use the output mapping $2q_i - 1$ where q_i is the final state, then the 2-automaton in Figure 4.1 generates \mathfrak{G} . The relationship between multiplicative functions and finite automata will be discussed in more detail in Chapter 5.

Write

$$W_m(z) = \sum_{k=0}^{m-1} \frac{z^{2^k}}{1 + z^{2^{k+1}}}.$$

Using the functional equation from Lemma 4.12, we have

$$G(z^{2^m}) = G(z) - \sum_{k=0}^{m-1} \frac{z^{2^k}}{1 + z^{2^{k+1}}}$$

so that

$$G(z^{2^m}) = G(z) - W_m(z). \tag{4.2}$$

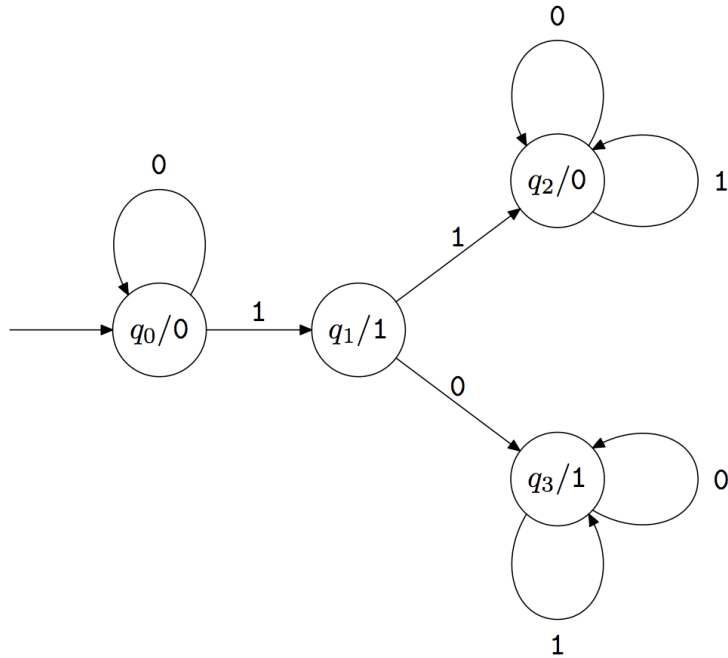


Figure 4.1: The 2-automaton that produces the sequence \mathfrak{G} .

Proposition 4.14. *If $|z| < 1$, then the generating function of \mathfrak{G} has the closed form*

$$G(z) = \sum_{k \geq 0} \frac{z^{2^k}}{1 + z^{2^{k+1}}}.$$

Proof. Take the limit as $m \rightarrow \infty$ in relation (4.2). □

Note that

$$\gamma = G\left(\frac{1}{2}\right) = \lim_{m \rightarrow \infty} \left[G(2^{-2^m}) + W_m\left(\frac{1}{2}\right) \right] = \sum_{k \geq 0} \frac{1}{2^{2^k} + 2^{-2^k}}.$$

Theorem 4.15. *The function $G(z)$ is transcendental over $\mathbb{C}(z)$.*

Proof. Towards a contradiction, suppose that $G(z)$ is algebraic over $\mathbb{C}[z]$; that is, there is an $n \geq 1$ and rational functions $q_0(z), q_1(z), \dots, q_{n-1}(z)$ such that

$$G(z)^n + q_{n-1}(z)G(z)^{n-1} + \dots + q_0(z) = 0 \quad (|z| < 1).$$

Choose n minimally. By the functional equation of Lemma 4.12 we obtain

$$\sum_{k=0}^n q_k(z^2) \left[G(z) - \frac{z}{1+z^2} \right]^k = 0 \quad (|z| < 1),$$

so that for $|z| < 1$

$$\begin{aligned} P(z) &:= \sum_{k=0}^n q_k(z^2)[(1+z^2)G(z) - z]^k [1+z^2]^{n-k} \\ &= \sum_{k=0}^n q_k(z^2)[1+z^2]^{n-k} \sum_{j=0}^k \binom{k}{j} (1+z^2)^j G(z)^j (-z)^{k-j} = 0. \end{aligned}$$

Thus

$$Q(z) := q_n(z)P(z) - (1+z^2)^n q_n(z^2) \sum_{k=0}^n q_k(z)G(z)^k = 0. \quad (4.3)$$

Inspection of $Q(z)$ gives the $k = n$ term as

$$q_n(z)q_n(z^2) \left(\sum_{j=0}^n \binom{n}{j} (1+z^2)^j G(z)^j (-z)^{n-j} - (1+z^2)^n G(z)^n \right).$$

The coefficient of $G^n(x)$ is given when we set $j = n$ in the preceding line, and is

$$q_n(z)q_n(z^2) \left(\binom{n}{n} (1+z^2)^n G(z)^n (-z)^{n-n} - (1+z^2)^n G(z)^n \right) = 0.$$

Hence (4.3) defines polynomials $h_1(z), \dots, h_{n-1}(z)$ such that

$$Q(z) = \sum_{k=0}^{n-1} h_k(z)G(z)^k = 0.$$

The minimality of n implies that $h_k(z) = 0$ for $k = 0, \dots, n-1$.

Let us now determine $h_k(z)$ using the definition of $Q(z)$ from (4.3). We have

$$\begin{aligned} Q(z) &= \sum_{k=0}^{n-1} h_k(z)G(z)^k \\ &= \sum_{k=0}^n \left\{ \sum_{j=0}^k \binom{k}{j} q_n(z)q_k(z^2)(1+z^2)^{n-k+j} G(z)^j (-z)^{k-j} \right. \\ &\quad \left. - (1+z^2)^n q_n(z^2)q_k(z)G(z)^k \right\} \end{aligned}$$

From here we can read off the coefficient of $G(z)^m$ as

$$h_m(z) = \sum_{k=m}^n \binom{k}{m} q_k(z^2)(1+z^2)^{n-k+m} (-z)^{k-m} - (1+z^2)^n q_m(z).$$

Since $h_{n-1}(z) = 0$, we have

$$\sum_{k=n-1}^n \binom{k}{n-1} q_k(z^2)(1+z^2)^{n-k+n-1}(-z)^{k-(n-1)} = (1+z^2)^n q_{n-1}(z).$$

Hence,

$$q_{n-1}(z^2)(1+z^2)^n - nz(1+z^2)^{n-1} = (1+z^2)^n q_{n-1}(z),$$

so that we may focus on the equation

$$(1+z^2)q_{n-1}(z^2) - nz = (1+z^2)q_{n-1}(z).$$

Write $q_{n-1}(z) = \frac{a(z)}{b(z)}$, where $a(z)$ and $b(z)$ are polynomials with no common factors. Substituting and clearing denominators, we obtain

$$(1+z^2)a(z^2)b(z) - nzb(z)b(z^2) = (1+z^2)a(z)b(z^2). \quad (4.4)$$

Using simple divisibility rules, (4.4) gives the following two conditions:

$$(i) \frac{b(z)}{G(x)} \Big| (1+z^2) \frac{b(z^2)}{G(x)} \quad \text{and} \quad (ii) \frac{b(z^2)}{G(x)} \Big| (1+z^2) \frac{b(z)}{G(x)},$$

where $G(x) = \gcd(b(z), b(z^2))$. Recall that $(1+z^2) = (1+iz)(1-iz)$.

A side note on determining properties of $b(z)$: condition (ii) indicates the degree relationship, $2 \deg b(z) \leq 2 + \deg b(z)$, and equation (4.4) implies that $(z^2+1) \mid b(z)b(z^2)$ which gives $2 \leq 3 \deg b(z)$. Together this yields a degree condition on $b(z)$ of $1 \leq \deg b(z) \leq 2$, since the degree must be a positive integer. In light of conditions (i) and (ii), we have $\deg b(z) = 2$.

Now conditions (i) and (ii) imply that either

$$(1+iz) \mid b(z) \quad \text{and} \quad (1-iz) \mid b(z^2), \quad (4.5)$$

or

$$(1-iz) \mid b(z) \quad \text{and} \quad (1+iz) \mid b(z^2). \quad (4.6)$$

Given the above conditions, we have two options for $b(z)$:

$$\text{condition (4.5)} \implies b(z) = (z+1)(z-i),$$

or

$$\text{condition (4.6)} \implies b(z) = (z+1)(z+i).$$

Let us assume that condition (4.5) holds and that $b(z) = (z + 1)(z - i)$. Then (4.4) becomes

$$(1 + z^2)(z + 1)(z - i)a(z^2) - nz(z + 1)(z - i)(z^2 + 1)(z^2 - i) = (1 + z^2)(z^2 + 1)(z^2 - i)a(z).$$

Removing common factors, the last equation becomes

$$(z + 1)a(z^2) - nz(z + 1)(z^2 - i) = (z + i)(z^2 - i)a(z). \quad (4.7)$$

Equation (4.7) implies that $(z + 1) \mid a(z)$, and so $(z^2 + 1) \mid a(z^2)$. Thus there exist $k(z), l(z)$ such that

$$a(z) = k(z)(z + 1) \quad \text{and} \quad a(z^2) = l(z)(z^2 + 1) = l(z)(z + i)(z - i).$$

Equation (4.7) becomes

$$(z + 1)l(z)(z + i)(z - i) - nz(z + 1)(z^2 - i) = (z + i)(z^2 - i)k(z)(z + 1),$$

implying that

$$(z + i) \mid nz(z + 1)(z^2 - 1),$$

which is not possible. Hence it must be the case that $b(z) \neq (z + 1)(z - i)$, and so condition (4.5) is not possible.

It must now be the case that condition (4.6) holds, that $b(z) = (z + 1)(z + i)$. Then (4.4) becomes

$$(1 + z^2)(z + 1)(z + i)a(z^2) - nz(z + 1)(z + i)(z^2 + 1)(z^2 + i) = (1 + z^2)(z^2 + 1)(z^2 + i)a(z).$$

Removal of common factors yields

$$(z + 1)a(z^2) - nz(z + 1)(z^2 + i) = (z - i)(z^2 + i)a(z). \quad (4.8)$$

Similar to the previous case, (4.8) implies that $(z + 1) \mid a(z)$, and so $(z^2 + 1) \mid a(z^2)$. Thus there exist $k(z), l(z)$ such that

$$a(z) = k(z)(z + 1) \quad \text{and} \quad a(z^2) = l(z)(z^2 + 1) = l(z)(z + i)(z - i).$$

Equation (4.8) becomes

$$(z + 1)l(z)(z + i)(z - i) - nz(z + 1)(z^2 + i) = (z - i)(z^2 + i)k(z)(z + 1),$$

implying that

$$(z + i) \mid nz(z + 1)(z^2 + i),$$

which is impossible. Since one of conditions (4.5) or (4.6) must hold, we arrive at our final contradiction, and the theorem is proved. \square

We proceed to show that γ is transcendental over \mathbb{Q} , by again using Mahler's method. Lemma 4.12, Theorem 4.15, and Theorem 3.4 give our next theorem.

Theorem 4.16. *The Gaussian Liouville number*

$$\gamma = G\left(\frac{1}{2}\right) = \sum_{k \geq 0} \frac{1}{2^{2^k} + 2^{-2^k}}$$

is transcendental over \mathbb{Q} .

Proof. Lemma 4.12 gives the functional equation

$$G(z^2) = \frac{(1 + z^2)G(z) - z}{1 + z^2},$$

so that, in the language of Theorem 3.4, we have

$$A(u) = (1 + z^2)u - z \quad \text{and} \quad B(u) = 1 + z^2,$$

$m = 1 < 2 = d$, and $a_i(z), b_i(z) \in \mathbf{I}_K[z]$. Since $B(u)$ is a constant polynomial in u , we have

$$\Delta(z) := \text{Res}(A, B) = 1 + z^2.$$

Set $\alpha = \frac{1}{2}$. It is immediate that $\alpha = \frac{1}{2}$ is algebraic, $0 < |\alpha| = \frac{1}{2} < \min\{1, R\}$ ($R = 1$), and

$$\Delta(\alpha^{2^k}) = \Delta(2^{-2^k}) = 1 + 2^{-2^{k+1}} \neq 0 \quad (k \geq 0).$$

Since $G(z)$ is not algebraic over $\mathbb{C}[z]$ (as supplied by Theorem 4.15), applying Theorem 3.4, we have that γ is transcendental over \mathbb{Q} . \square

4.2.3 Transcendence related to character-like functions

A *character-like function* f associated to p is a completely multiplicative function from \mathbb{N} to $\{-1, 1\}$ defined by $f_1 = 1$, $f_p = \pm 1$ (your choice), and $f_{kp+i} = f_i$. As an example, the completely multiplicative function defined by

$$f_n = \begin{cases} \pm 1 & \text{if } n = p \\ \left(\frac{n}{p}\right) & \text{if } p \nmid n, \end{cases}$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol modulo p , is a character-like function.

If we let f be a character-like function associated to p , then for $F(z) = \sum_{n \geq 1} f_n z^n$ the generating function of the sequence $\mathfrak{F} := (f_n)$ we have a lemma similar to the previous subsections.

Lemma 4.17. *The generating function $F(z)$ of the sequence \mathfrak{F} satisfies the functional equation*

$$F(z) = f_p F(z^p) + \frac{\Phi(z)}{1 - z^p},$$

where $\Phi(z) = \sum_{i=1}^{p-1} f_i z^i$.

Proof. We have

$$\begin{aligned} F(z) &= \sum_{k \geq 0} \sum_{i=1}^{p-1} f_{pk+i} z^{pk+i} + \sum_{k \geq 1} f_{pk} z^{pk} \\ &= \sum_{i=1}^{p-1} f_i z^i \sum_{k \geq 0} z^{pk} + f_p \sum_{k \geq 1} f_k z^{pk} \\ &= \frac{\sum_{i=1}^{p-1} f_i z^i}{1 - z^p} + f_p F(z^p). \quad \square \end{aligned}$$

Theorem 4.18. *The function $F(z)$ is transcendental.*

Proof. Note that since $z = 1$ is a root of $1 - z^p$ and $\Phi(1) = \sum_{i=1}^{p-1} (i/p) = 0$, the rational function $\frac{\Phi(z)}{1 - z^p}$ may be written as the ratio of two polynomials, both of which have degree strictly less than p . Hence an application of Theorem 3.13 gives the result. \square

Using the functional equation, we yield

$$F(z^p) = f_p F(z) - f_p \frac{\Phi(z)}{1 - z^p}.$$

We can build a series for the number as before. The functional equation gives

$$\begin{aligned} F(z^{p^m}) &= f_p F(z^{p^{m-1}}) - f_p \frac{\Phi(z^{p^{m-1}})}{1 - z^{p^m}} \\ &= f_p^2 F(z^{p^{m-2}}) - f_p^2 \frac{\Phi(z^{p^{m-2}})}{1 - z^{p^{m-1}}} - f_p \frac{\Phi(z^{p^{m-1}})}{1 - z^{p^m}} \\ &= f_p^m F(z) - \sum_{k=1}^m f_p^k \frac{\Phi(z^{p^{m-k}})}{1 - z^{p^{m-k+1}}}, \end{aligned}$$

which when rearranged leads to

$$F(z) = f_p^m F(z^{p^m}) + \sum_{k=1}^m f_p^{m-k} \frac{\Phi(z^{p^{m-k}})}{1 - z^{p^{m-k+1}}}.$$

We change the index and set

$$V_m(z) := \sum_{k=0}^{m-1} f_p^k \frac{\Phi(z^{p^k})}{1 - z^{p^{k+1}}}$$

to give

$$F(z) = f_p^m F(z^{p^m}) + V_m(z).$$

Proposition 4.19. *The generating function $F(z)$ has the closed form*

$$F(z) = \sum_{k \geq 0} f_p^k \frac{\Phi(z^{p^k})}{1 - z^{p^{k+1}}}.$$

Proof. Take the limit as $m \rightarrow \infty$ in the equation $F(z) = f_p^m F(z^{p^m}) + V_m(z)$. □

At $z = 1/2$ we have

$$F\left(\frac{1}{2}\right) = \lim_{m \rightarrow \infty} \left[f_p^m F(2^{-p^m}) + V_m\left(\frac{1}{2}\right) \right] = \sum_{k \geq 0} \frac{f_p^k \Phi(2^{-p^k})}{1 - 2^{-p^{k+1}}}.$$

Theorem 4.20. *For each odd prime p , the number $\varphi_p := F\left(\frac{1}{2}\right)$ is transcendental.*

Proof. Lemma 4.17 gives

$$F(z^p) = \frac{f_p(1 - z^p)F(z) - f_p\Phi(z)}{1 - z^p},$$

where $\Phi(z) = \sum_{i=1}^{p-1} f_i z^i$. Similar to the specific case of the previous section, using the language of Mahler's Theorem, we have

$$A(u) = f_p(1 - z^p)u - f_p\Phi(z) \quad \text{and} \quad B(u) = 1 - z^p,$$

$m = 1 < p = d$, and $a_i(z), b_i(z) \in \mathbf{I}_{\mathbb{C}}[z]$. Again, $B(u)$ is a constant polynomial in u , so that

$$\Delta(z) := \text{Res}(A, B) = 1 - z^p.$$

Set $\alpha = \frac{1}{2}$; $\alpha = \frac{1}{2}$ is algebraic, $0 < |\alpha| = \frac{1}{2} < \min\{1, R\}$ ($R = 1$), and

$$\Delta(\alpha^{p^k}) = \Delta(2^{-p^k}) = 1 - 2^{-p^{k+1}} \neq 0 \quad (k \geq 0).$$

Theorem 4.18 gives that $F(z)$ is not algebraic over $\mathbb{C}[z]$ and we may apply Theorem 3.4 to give the desired result. □

Remark 4.21. Mahler's Theorem tells us that the values of the power series $G(z)$ and $F(z)$ are transcendental for any algebraic value of z within their radii of convergence. The special value of $z = \frac{1}{2}$ is focused on only for its relation to the sequences \mathfrak{G} and \mathfrak{F} .

Remark 4.22. There is a very powerful theorem of Adamczewski and Bugeaud [1] which states that *if $\beta = \sum_n a(n)b^{-n}$ is a base b automatic irrational, then β is transcendental.* Their proof uses the Schmidt Subspace Theorem, and avoids Mahler theory. Mahler's theory still yields a more complete result, as it gives transcendence of a series at not just the values $\frac{1}{b}$ for $b \geq 2$, but for all algebraic values in the region of convergence. Indeed, it is not known whether a Mahler-theoretic proof of Adamczewski and Bugeaud's result is possible.

Chapter 5

(Non)Automaticity

5.1 Automaticity

The general theory of automaticity is not a well-known area to many number theorists. This section contains background results for those who are not well acquainted with this theory. Unless otherwise noted, the definitions, theorems, and proofs in this section have been taken from Allouche and Shallit [6].

Definition 5.1. A k -deterministic finite automaton with output (k -DFAO) is defined to be a 6-tuple

$$M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$$

where Q is a finite set of states, $\Sigma_k = \{0, 1, \dots, k-1\}$ is the finite input alphabet, $\delta : Q \times \Sigma_k \rightarrow Q$ is the transition function, $q_0 \in Q$ is the initial state, Δ is the output alphabet, and $\tau : Q \rightarrow \Delta$ is the output function.

Definition 5.2. We say that the sequence $(a(n))_{n \geq 0}$ over a finite alphabet Δ is k -automatic if there exists a k -DFAO $M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$ such that $a(n) = \tau(\delta(q_0, w))$ for all $n \geq 0$ and all w with $[w]_k = n$.

While the definition of a k -DFAO has a computational appeal, from a theoretical point of view, for the types of results we are interested in, the setting of Definition 5.2 is more tractable.

Theorem 5.3. *The sequence $(a(n))_{n \geq 0}$ is k -automatic if and only if there exists a k -DFAO M such that $a(n) = \tau(\delta(q_0, (n)_k))$ for all $n \geq 0$. Moreover, we may choose M such that*

$$\delta(q_0, 0) = q_0.$$

Proof. The first direction is clear from the definition. For the other, let $M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$ and define $M' = (Q', \Sigma_k, \delta', q'_0, \Delta, \tau')$ by

$$\begin{aligned} Q' &= Q \cup \{q'_0\}, \\ \delta'(q, b) &= \delta(q, b) \quad \text{for all } q \in Q, b \in \Sigma_k, \\ \delta'(q'_0, b) &= \begin{cases} \delta(q_0, b) & \text{if } b \neq 0, \\ q'_0 & \text{if } b = 0, \end{cases} \\ \tau'(q'_0) &= \tau(q_0), \\ t'(q) &= \tau(q) \quad \text{for all } q \in Q. \end{aligned}$$

For $n \neq 0$, we have

$$\delta'(q'_0, 0^i(n)_k) = \delta'(q_0, (n)_k) = \delta(q_0, (n)_k).$$

Hence

$$\tau'(\delta'(q'_0, 0^i(n)_k)) = \tau(\delta(q_0, (n)_k)). \quad \square$$

If we allow the k -DFAO to take the base k expansion of an integer n as input, starting with the least significant digit, we have the following theorem (see Theorem 5.2.3 of [6]).

Theorem 5.4. *The following three conditions are equivalent:*

- (i) $(a(n))_{n \geq 0}$ is a k -automatic sequence.
- (ii) There exists a k -DFAO $M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$ such that $a(n) = \tau(\delta(q_0, w^R))$ for all $n \geq 0$ and all finite words w from the alphabet Σ_k such that $[w]_k = n$.
- (iii) There exists a k -DFAO $M' = (Q', \Sigma_k, \delta', q'_0, \Delta, \tau')$ such that $a(n) = \tau(\delta(q_0, (n)_k^R))$ for all $n \geq 0$.

For those of us are not comfortable with the k -DFAO terminology, the above definition of a k -automatic sequence can be confusing. Fortunately, there is an equivalent way to view this concept which, in our opinion, is more intuitive.

Definition 5.5. Let $\mathbf{a} = (a(n))_{n \geq 1}$ be a sequence with values from a finite set. Define the k -kernel of \mathbf{a} as the set

$$K_k(\mathbf{a}) = \{(a(k^l n + r))_{n \geq 0} : l \geq 0 \text{ and } 0 \leq r < k^l\}.$$

Theorem 5.6. *Let $k \geq 2$. The sequence $\mathbf{a} = (a(n))_{n \geq 0}$ is k -automatic if and only if $K_k(\mathbf{a})$ is finite.*

Proof. Let $k \geq 2$. Suppose that $\mathbf{a} = (a(n))_{n \geq 0}$ is k -automatic sequence. Then it follows from Theorem 5.4 that there is a k -DFAO $M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$ such that

$$a(n) = \tau(\delta(q_0, (n)_k^R 0^t)) \quad \forall t \geq 0.$$

Now let $q = \delta(q_0, w^R)$ where $|w| = i$ and $[w]_k = j$. Since

$$(k^i n + j)_k = (n)_k w$$

expect possibly when $n = 0$, it follows that, for $n > 0$, we have

$$\delta(q_0, (k^i n_j)_k^R) = \delta(\delta(q_0, w^R), (n)_k^R) = \delta(q, (n)_k^R).$$

In the case when $n = 0$ we have

$$(k^i n + j)_k = (j)_k,$$

and $w = 0^t(j)_k$ for some $t \geq 0$. Then

$$\delta(q_0, (k^i n_j)_k^R) = \delta(q_0, (j)_k^R) = \delta(q_0, (j)_k^R 0^t) = \delta(q_0, w^R) = q = \delta(q, (0)_k^R).$$

It follows that the subsequence $(a(k^i n + j))_{n \geq 0}$ is generated by the k -DFAO $(Q, \Sigma_k, \delta, q, \Delta, \tau)$. Since there are only finitely many choices for q , the finiteness of $K_k(\mathbf{a})$ follows.

Now suppose that $K_k(\mathbf{a})$ is finite. Then the set of finite words on Σ_k is partitioned into a finite number of disjoint equivalence classes under the equivalence relation

$$w \equiv x \quad \text{if and only if} \quad a(k^{|w|}n + [w]_k) = a(k^{|x|}n + [x]_k)$$

for all $n \geq 0$.

We make a k -DFAO as follows:

$$Q = \{[x] : x \text{ is a finite word on } \Sigma_k\}, \quad \delta([x], b) = [bx], \quad \tau([w]) = a([w]_k), \quad q_0 = [\varepsilon],$$

where $[x]$ is the equivalence class containing x . We need to see that this definition is meaningful, that is, if $[x] = [w]$, then $\delta([x], b) = \delta([w], b)$ and $\tau([x]) = \tau([w])$. For the first, we need $[bx] = [bw]$. Now if $[x] = [w]$ then

$$a(k^{|w|}n + [w]_k) = a(k^{|x|}n + [x]_k)$$

for all $n \geq 0$. Setting $n = km + b$ it follows that

$$a(k^{|aw|}m + [aw]_k) = a(k^{|ax|}m + [ax]_k)$$

for all $m \geq 0$.

For the second assertion we need to see that if

$$a(k^{|w|}n + [w]_k) = a(k^{|x|}n + [x]_k)$$

then

$$a([w]_k) = a([x]_k).$$

To do this, set $n = 0$. We now claim that $\tau(\delta(q_0, w^R)) = a([w]_k)$ for all finite words w on Σ_k . By induction on $|w|$ we have $\delta(q_0, w^R) = [w]$. By the definition of τ the result follows. \square

Connecting automaticity to transcendence, we have the following beautiful and crucial theorem of Christol et al. which is essential to the results of this chapter. The proof given here is our translation from the original French [23].

Theorem 5.7 (Christol et al. [23]). *Let Σ be a finite nonempty alphabet, $\mathbf{t} = (t_n) \in \Sigma^{\mathbb{N}}$, and p be a prime number. Then the sequence \mathbf{t} is p -automatic if and only if there exists a field K of characteristic p and an injection $\alpha : \Sigma \rightarrow K$ such that $\alpha(\mathbf{t}) = (\alpha(t_n))$ is algebraic over $K(X)$.*

Proof. Suppose that $\mathbf{t} \in \Sigma^{\mathbb{N}}$ is a p -automatic sequence. Also, let K be a finite field of characteristic p having at least as many elements as Σ . Thus Σ can be embedded in K . After renaming the elements of Σ , we may consider $\mathbf{t} \in K^{\mathbb{N}}$. We show that \mathbf{t} is algebraic over $K(X)$.

Denote the number of elements of K by $q := p^s$.

For $r \in [q]$, we consider the application $A_r : K[[X]] \rightarrow K[[X]]$ defined by

$$A_r \left(\sum_n v_n X^n \right) = \sum_n v_{qn+r} X^n.$$

Let \mathcal{N} be the semigroup generated by the identity and the A_r under composition. For all $v = \sum_n v_n X^n \in K[[X]]$ we associate its orbit

$$\mathcal{N}(v) = \{A(v) : A \in \mathcal{N}\}.$$

By Theorem 5.6, we know that $\mathcal{N}(v)$ is finite if and only if v is q -automatic, that is to say p -automatic (see [41, Proposition 3.3, p. 107]). Thus $\mathcal{N}(\mathfrak{t})$ is finite.

Let E be the vector space over $K(X)$ generated by $\mathcal{N}(\mathfrak{t})$ and F the vector space over $K(X)$ generated by the g^q , where $g \in \mathcal{N}(\mathfrak{t})$. We show initially that $E = F$. Indeed, if g_1, g_2, \dots, g_N are a basis of E , then for all $g \in E$ we have

$$g = \sum_{k=1}^N c_k g_k$$

thus

$$g^q = \sum_{k=1}^N c_k^q g_k^q$$

which shows that $g_1^q, g_2^q, \dots, g_N^q$ is a system of generators of F . Thus

$$\dim F \leq \dim E \leq \text{card} \mathcal{N}(\mathfrak{t}).$$

In addition, for all $g \in \mathcal{N}(\mathfrak{t})$, we have

$$g = \sum_{r=0}^{q-1} X^r (A_r(g))^q.$$

However $(A_r(g))^q \in (\mathcal{N}(\mathfrak{t}))^q \subset F$, thus $E \subset F$, so that the preceding inequality gives $E = F$.

Now let G be the vector space over $K(X)$ generated by products of the type

$$\prod_{g \in \mathcal{N}(\mathfrak{t})} g^{\beta(g)},$$

where $\beta : \mathcal{N}(\mathfrak{t}) \rightarrow \mathbb{N}$ is not identically zero. Then $\mathfrak{t} \in G$, where G is a ring with $\mathfrak{t}G \subset G$. However, a classical result [69, p. 2] gives the implication

$$\dim G < \infty \quad \Rightarrow \quad \mathfrak{t} \text{ is algebraic}$$

and the degree of \mathfrak{t} is bounded by $\dim G$. Thus to establish that \mathfrak{t} is algebraic it suffices to show that G has finite dimension.

Let $g \in \mathcal{N}(\mathfrak{t})$. Then since $E = F$, we have that g^q is a linear combination of $h \in \mathcal{N}(\mathfrak{t})$ with coefficients in $K(X)$. Consequently, G is generated by the products

$$\prod_{g \in \mathcal{N}(\mathfrak{t})} g^{\beta(g)} \quad \text{where } \beta(g) < q.$$

It follows that

$$\dim G \leq q^{\text{card } \mathcal{N}(\mathbf{t})} - 1.$$

This gives the first implication of the theorem.

For the other direction let K be a finite field of characteristic p of $q = p^s$ elements, $\mathbf{f} = (f_n)$ a sequence algebraic over $K(X)$, α an injection of Σ into K , and $\mathbf{t} = (t_n) = (\alpha^{-1}f_n)$. We show below that \mathbf{t} is q -automatic, and hence by [41] \mathbf{t} is p -automatic, which gives that f is q -automatic.

We use the same notations as above. As we have already seen, it suffices to show that $\mathcal{N}(f)$ is finite.

Since f is algebraic over $K(X)$, the vector space generated by the f^n ($n \in \mathbb{N}$) has finite dimension over $K(X)$, and thus there exist a_0, \dots, a_k in $K[X]$ not all zero such that

$$\sum_{i=0}^k a_i f^{q^i} = 0.$$

Let j be the least integer for which there is a relation of the preceding type with $a_j \neq 0$. We show that $j = 0$. Indeed, since

$$a_j = \sum_{r=0}^{q-1} X^r (A_r(a_j))^q,$$

there exists an r with $A_r(a_j) \neq 0$.

From the relation $\sum_{i=j}^k a_i f^{q^i} = 0$, we deduce that $\sum_{i=j}^k A_r(a_i f^{q^i}) = 0$, and taking into account that for g and h in $K(X)$ we have $A_r(g^q h) = g A_r(h)$, and supposing $j \neq 0$, we have

$$\sum_{i=j}^k A_r(a_i) f^{q^{i-1}} = 0$$

which is a relation of the preceding type where the coefficient of $f^{q^{j-1}}$ is different from 0, which is in contradiction to the hypothesis made on j .

Thus we have the relation

$$\sum_{i=0}^k a_i f^{q^i} = 0 \quad \text{with} \quad a_0 \neq 0.$$

Writing $g = f/a_0$, we have

$$g = \sum_{i=1}^k b_i g^{q^i} \quad \text{where} \quad b_i = -a_i a_0^{q^i - 2} \in K[X].$$

Let $N = \sup(\deg a_0, \sup_{i=1, \dots, k} \deg b_i)$, and let H be the set of $h \in K[[X]]$ of the form

$$h = \sum_{i=0}^k c_i g^{q^i}, \quad c_i \in K[X], \quad \deg c_i \leq N,$$

H is a finite set, and $f = a_0 g$ belongs to h . It suffices to show that H is stable under applications A_r .

However, if h belongs to H , we have

$$A_r(h) = A_r\left(c_0 g + \sum_{i=1}^k c_i g^{q^i}\right) = A_r\left(\sum_{i=1}^k (c_0 b_i + c_i) g^{q^i}\right) = \sum_{i=1}^k A_r(c_0 b_i + c_i) g^{q^{i-1}}.$$

Since $\deg(c_0 b_i + c_i) \leq 2N$, we have that $\deg A_r(c_0 b_i + c_i) \leq 2N/q \leq N$ and consequently $A_r(h)$ belongs to H , which finishes the proof of the theorem. \square

For the remainder of this chapter, we rely heavily on the following corollary of the previous theorem of Christol et al.

Corollary 5.8. *Let \mathbb{F}_p be a finite field and $(a(n))_{n \geq 0}$ be a sequence with values in \mathbb{F}_p . Then, the sequence $(a(n))_{n \geq 0}$ is p -automatic if and only if the formal power series $\sum_{n \geq 0} a(n) X^n$ is algebraic over $\mathbb{F}_p(X)$.*

5.2 (Non)Automaticity of arithmetic functions

In Banks, Luca, and Shparlinski [8], it is shown that the series

$$\sum_{n \geq 1} f(n) X^n \in \mathbb{Z}[[X]] \tag{5.1}$$

is not a rational function with coefficients in \mathbb{Z} when f is taken to be any of the number-theoretic functions

$$\varphi, \tau, \sigma, \lambda, \mu, \omega, \Omega, p, \text{ or } \rho. \tag{5.2}$$

Here $\varphi(n)$, the Euler totient function, is the number of positive integers $m \leq n$ with $\gcd(m, n) = 1$, $\tau(n)$ is the number of positive integer divisors of n , $\sigma(n)$ is the sum of those divisors, $\omega(n)$ is the number of distinct prime divisors of n , $\Omega(n)$ is the number of total prime divisors of n , $\lambda(n) = (-1)^{\Omega(n)}$ is Liouville's function, $\mu(n)$ is the Möbius function

defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } k^2 \mid n \text{ for some } k \geq 2, \\ (-1)^{\omega(n)} & \text{if } k^2 \nmid n \text{ for all } k \geq 2, \end{cases}$$

$p(n)$ is the n -th prime number, and $\rho(n) = 2^{\omega(n)}$ counts the number of square-free positive divisors of n .

In the course of this investigation we give, or give reference to, results showing that the series $\sum_{n \geq 1} f(n)X^n \in \mathbb{Z}[[X]]$ is transcendental over $\mathbb{Z}(X)$, for all of the functions f in (5.2). In most cases, the stronger result of transcendence of the series in $\mathbb{F}_p[[X]]$ over $\mathbb{F}_p(X)$ is shown. To get at these stronger results we rely upon Corollary 5.8.

Since any algebraic relation in $\mathbb{Z}(X)$ is an algebraic relation in $\mathbb{F}_p(X)$, we have

Lemma 5.9. *Let p be a prime. If a series $F(X) \in \mathbb{Z}[[X]]$ and its termwise reduction $\bar{F}(X) \in \mathbb{F}_p[[X]]$ is transcendental over $\mathbb{F}_p(X)$, then $F(X) \in \mathbb{Z}[[X]]$ is transcendental over $\mathbb{Z}(X)$.*

Between Allouche [3] and Yazdani [98] we have that for any prime p , the series (5.1) is transcendental over $\mathbb{F}_p(X)$ (and so over $\mathbb{Z}(X)$ by the lemma) for $f = \varphi, \tau_k, \sigma_k$, and μ . Recall that

$$\tau_k(n) := \#\{(a_1, a_2, \dots, a_k) : a_1 a_2 \cdots a_k = n, a_i \in \mathbb{N} \text{ for } i = 1, \dots, k\}$$

and $\sigma_k(n)$ is the sum of the k th powers of the divisors of n (note that $\tau_2(n) = \tau(n)$ and $\sigma_1(n) = \sigma(n)$). In the previous chapter we showed that the series (5.1) is transcendental over $\mathbb{Z}(X)$ for any completely multiplicative function $f : \mathbb{N} \rightarrow \{-1, 1\}$ that is not identically 1; this includes $f = \lambda$. We summarize in the following two theorems.

Theorem 5.10 (Allouche [3], Yazdani [98]). *The series $\sum_{n \geq 1} f(n)X^n$ is transcendental over $\mathbb{F}_p(X)$ for $f = (g \pmod v)$ with $g = \varphi, \tau_m, \sigma_m$, and μ where $m \geq 1$ and $v \geq 2$.*

Theorem 5.11. *The series $\sum_{n \geq 1} f(n)X^n$ is transcendental over $\mathbb{Z}(X)$ for any nontrivial completely multiplicative function taking values in $\{-1, 1\}$ (this includes $f = \lambda$).*

In Section 5.3, answering a question of Yazdani [98], we give the main result of this chapter is an improvement of Theorem 5.11.

Theorem 5.12. *Liouville's function λ is not k -automatic for any $k \geq 2$, and hence $\sum_{n \geq 1} \lambda(n)X^n \in \mathbb{F}_p[[X]]$ is transcendental over $\mathbb{F}_p(X)$ for all $p > 2$.*

We can use Theorem 5.12 to prove a similar result for $\Omega(n)$. We also use the following direct consequence of the definition of automaticity.

Remark 5.13. Let $t : \mathbb{N} \rightarrow Y$ and $\Phi : Y \rightarrow Z$ be mappings. If $(t(n))_{n \geq 1}$ is k -automatic for some $k \geq 2$, then $(\Phi(t(n)))_{n \geq 1}$ is also k -automatic.

Note that the values of $\Omega(n)$ viewed modulo 2 satisfy

$$(\Omega(n) \bmod 2) = \frac{1 - \lambda(n)}{2}.$$

Using this relationship, Remark 5.13 and Theorem 5.12 give the following corollary.

Corollary 5.14. *The function $(\Omega(n) \bmod 2)$ is not 2-automatic. Furthermore, the series $\sum_{n \geq 1} \Omega(n)X^n$ is transcendental over both $\mathbb{F}_2(X)$, and $\mathbb{Z}(X)$.*

Ritchie [86] showed that the characteristic function of the squares is not 2-automatic. Combining this result with Remark 5.13 gives a nice corollary regarding $\tau(n)$.

Corollary 5.15. *The sequence $(\tau(n) \bmod 2)$ is not 2-automatic. Hence $\sum_{n \geq 1} \tau(n)X^n$ is transcendental over both $\mathbb{F}_2(X)$ and $\mathbb{Z}(X)$.*

Proof. The function $\tau(n)$ taken modulo 2 is the characteristic function of the squares. \square

One of the nicest results in this area is that of Hartmanis and Shank on the non-automaticity of the characteristic function of the primes.

Theorem 5.16 (Hartmanis and Shank [56]). *The characteristic function of the primes, χ_P , is not k -automatic for any $k \geq 2$.*

In Section 5.3, we give new (short and analytic) proofs of Theorem 5.16, as well as its extension to all prime powers, and Corollary 5.14. Many other functions, such as ρ , are also considered in this section. In Section 5.4, we also address unbounded multiplicative functions using the generalization of k -automatic sequences to k -regular sequences.

The differences in transcendence over $\mathbb{Z}(X)$ and $\mathbb{F}_p(X)$ are quite pronounced. Theorem 5.11 gives transcendence over $\mathbb{Z}(X)$ of a very large class of functions, many of which are k -automatic for some $k \geq 2$ and hence their reduction over some finite field is algebraic. For those $(f(n))_{n \geq 0}$ that are automatic, one can use the theory of Mahler [73, 83] to give transcendence results regarding the values of the series $\sum_{n \geq 1} f(n)X^n \in \mathbb{Z}[[X]]$. For non-automatic sequences almost no progress has been made. For example, the following conjecture is still open.

Conjecture 5.17. *Let $f : \mathbb{N} \rightarrow \{-1, 1\}$ be a completely multiplicative function for which $f(p) = -1$ for at least one prime p . Then the number $\sum_{n \geq 1} f(n)2^{-n}$ is transcendental over \mathbb{Q} .*

Remark 5.18. As some support for this conjecture, we may focus on those sequences here which are automatic. By Theorem 4.8, all of the numbers described in Conjecture 5.17 are irrational. A theorem of Adamczewski and Bugeaud [1], implies that if $(f(n))_{n \geq 1}$ from Conjecture 5.17 is k -automatic for some $k \geq 2$, then the number $\sum_{n \geq 1} f(n)2^{-n}$ is transcendental over \mathbb{Q} .

5.3 Dirichlet series and (non)automaticity

We rely heavily on a theorem of Allouche, Mendès France, and Peyrière [5], and also on the details of its proof. Before preceding to this theorem, we need some additional properties of k -automatic sequences (see [5] for details).

Let $k \geq 2$ and $(u(n))_{n \geq 1}$ be a k -automatic sequence with values in \mathbb{C} . Then there exist an integer $t \geq 1$ and a sequence $(U_n)_{n \geq 1}$ with values in \mathbb{C}^t (which we denote as a column vector) as well as k matrices of size $t \times t$ denoted by A_1, A_2, \dots, A_k , with the property that each row of each A_i has exactly one entry equal to 1 and the rest equal to 0 (the fact that these are ones and zeros comes from the finiteness of the k -kernel of $(u(n))_{n \geq 1}$), such that the first component of the vector $(U_n)_{n \geq 1}$ is the sequence $(u(n))_{n \geq 1}$ and for each $i = 1, 2, \dots, k$, and for all $n \geq 1$, we have

$$U_{kn+i} = A_i U_n.$$

Theorem 5.19 (Allouche, Mendès France, and Peyrière, [5]). *Let $k \geq 2$ be an integer and let $(u(n))_{n \geq 0}$ be a k -automatic sequence with values in \mathbb{C} . Then the Dirichlet series $\sum_{n \geq 1} u(n)n^{-s}$ is the first component of a Dirichlet vector (i.e., a vector of Dirichlet series) $G(s)$, where G has an analytic continuation to a meromorphic function on the whole complex plane, whose poles (if any) are located on a finite number of left semi-lattices.*

Proof. We follow the proof in [5], but with some slight modifications. Define a Dirichlet vector $G(s)$ for $\Re s > 1$ by

$$G(s) = \sum_{n \geq 1} \frac{U_n}{n^s}.$$

Since $U_{kn+j} = A_j U_n$, we have

$$G(s) = \sum_{j=1}^{k-1} \sum_{n \geq 1} \frac{A_j U_n}{(kn+j)^s} + \sum_{n \geq 1} \frac{A_k U_n}{(kn)^s}.$$

Writing I as the $t \times t$ identity matrix, we have

$$\begin{aligned} (I - k^{-s} A_k)G(s) &= \sum_{j=1}^{k-1} \sum_{n \geq 1} \frac{A_j U_n}{(kn+j)^s} \\ &= \sum_{j=1}^k A_j \sum_{n \geq 1} k^{-s} n^{-s} U_n \left(1 + \frac{j}{kn}\right)^{-s} \\ &= \sum_{j=1}^k A_j \sum_{m \geq 0} \binom{s+m-1}{m} (-j)^m \frac{G(s+m)}{k^{s+m}}, \end{aligned}$$

and so

$$(I - k^{-s}(A_0 + A_1 + \cdots + A_k))G(s) = \sum_{j=1}^k A_j \sum_{m \geq 1} \binom{s+m-1}{m} (-j)^m \frac{G(s+m)}{k^{s+m}}. \quad (5.3)$$

Denote $\mathcal{A} := k^{-1} \sum_{j=1}^k A_j$ and by $\mathcal{M}(X)$ the transpose of the comatrix of $(\mathcal{A} - XI)$, so that

$$\mathcal{M}(X)(\mathcal{A} - XI) = (\mathcal{A} - XI)\mathcal{M}(X) = \det(\mathcal{A} - XI)I.$$

Multiplying (5.3) by $\mathcal{M}(k^{s-1})$, we have

$$\det(\mathcal{A} - k^{s-1}I)G(s) = -\mathcal{M}(k^{s-1}) \sum_{j=1}^k A_j \sum_{m \geq 1} \binom{s+m-1}{m} (-j)^m \frac{G(s+m)}{k^{s+m}}. \quad (5.4)$$

For a given $s \in \mathbb{C}$, the function $G(s+m)$ is bounded for m large enough. Thus the right-hand side of (5.4) converges for $\Re s > 0$, so that this gives a meromorphic continuation to $\Re s \in (0, 1]$ with possible poles at points s for which k^{s-1} is an eigenvalue of \mathcal{A} . If $\Re s \in (-1, 0]$, the right-hand side of (5.4) converges with the possible exception of those s for which k^s is an eigenvalue of \mathcal{A} . This gives a meromorphic continuation of G to this region with possible poles at points s for which either k^{s-1} or k^s is an eigenvalue of \mathcal{A} . Continuing this process gives an analytic continuation of G to a meromorphic function on all of \mathbb{C} with possible poles at points

$$s = \frac{\log \alpha}{\log k} + \frac{2\pi i}{\log k} m - l + 1,$$

where α is an eigenvalue of \mathcal{A} , $m \in \mathbb{Z}$, $l \in \mathbb{N}$ and \log is a branch of the complex logarithm. \square

Remark 5.20. A one-dimensional version of Theorem 5.19 for the Thue–Morse Dirichlet series was given by Allouche and Cohen [4]. It is also worth noting that such an infinite functional equation is classical for $\zeta(s)$, the Riemann zeta function, though it is much less deep than the usual functional equation for $\zeta(s)$.

The beauty of the above proof is in the details, which is why we have chosen to reproduce it here. Note that the possible poles are explicitly given, as is the meromorphic continuation. This leads to a few nice classifications regarding Dirichlet series.

Proposition 5.21. *If the Dirichlet series $\sum_{n \geq 1} f(n)n^{-s}$ is not meromorphically continuable to the whole complex plane then $(f(n))_{n \geq 1}$ is not k -automatic for any $k \geq 2$.*

Our first application of this is a new proof of the well-known result of Hartmanis and Shank about the non-automaticity of the characteristic function of the primes.

Proof of Theorem 5.16. In 1920, Landau and Walfisz [68] proved that the Dirichlet series $P(s) := \sum_p p^{-s}$ is not continuable past the line $\Re s = 0$. This is a consequence of the identity

$$P(s) = \sum_{n \geq 1} \frac{\mu(n)}{n} \log \zeta(ns).$$

Since $\zeta(s)$ has a pole at $s = 1$, this relationship shows that $s = 1/n$ is a singular point for all square-free positive integers n . This sequence limits to $s = 0$. Indeed, all points on the line $\Re s = 0$ are limit points of the poles of $P(s)$ (see [92, pages 215–216] for details) so that the line $\Re s = 0$ is a natural boundary for $P(s)$. \square

Minsky and Papert [77] were the first to address this question, showing that the characteristic function of the primes is not 2-automatic. Hartmanis and Shank [56] gave the complete result. Similar to our proof of Theorem 5.16, denoting by

$$\chi_{\Pi}(n) := \begin{cases} 1 & \text{if } n \text{ is a prime power} \\ 0 & \text{otherwise,} \end{cases}$$

and using the relationship

$$\sum_{n \geq 1} \frac{\chi_{\Pi}(n)}{n^s} = \sum_{k \geq 1} \sum_{n \geq 1} \frac{\mu(n)}{n} \log \zeta(kns),$$

we have the corresponding result for prime powers.

Proposition 5.22. *The sequence $(\chi_{\Pi}(n))_{n \geq 1}$ is not k -automatic for any $k \geq 2$.*

Proposition 5.23. *Define the function $r(n)$ by $2 \cdot r(n) = \rho(n)$. The sequence $(r(n) \bmod 2)$ is not 2-automatic; hence $\sum_{n \geq 1} \rho(n)X^n$ is transcendental over $\mathbb{Z}(X)$.*

Proof. This follows from the fact that $(r(n) \bmod 2) = \chi_{\Pi}(n)$ and an application of Proposition 5.22. \square

As alluded to, the proof of Theorem 5.19 reveals much in the way of details. Indeed, due to the explicit determination of the poles, we can provide a very useful classification, but first, a definition.

Definition 5.24. Denote by $R(a, b; T)$ the rectangular subset of \mathbb{C} defined by $\Re s \in [a, b]$ and $\Im s \in [0, T]$, by $N_{\infty}(F(s), R(a, b; T))$ the number of poles of $F(s)$ in $R(a, b; T)$, and by $N_0(F(s), R(a, b; T))$ the number of zeros of $F(s)$ in $R(a, b; T)$.

Proposition 5.25. *Let $k \geq 2$, $(f(n))_{n \geq 1}$ be a k -automatic sequence and let $F(s)$ denote the Dirichlet series with coefficients $(f(n))_{n \geq 1}$. If $a, b \in \mathbb{R}$ with $a < b$, then*

$$N_{\infty}(F(s), R(a, b; T)) = O(T).$$

Hence, if $G(s) = \sum_{n \geq 1} g(n)n^{-s}$ ($\Re s > \alpha$ for some $\alpha \in \mathbb{R}$) is meromorphically continuable to a region containing a rectangle $R(a, b, T)$ for which

$$\lim_{T \rightarrow \infty} \frac{1}{T} N_{\infty}(F(s), R(a, b, T)) = \infty,$$

then $(g(n))_{n \geq 1}$ is not k -automatic for any $k \geq 2$.

Proof. This is a direct consequence of the poles of F being located on a finite number of left semi-lattices. \square

From here on, we make systematic use of a classical result by von Mangoldt.

Theorem 5.26 (von Mangoldt [94]). *The number of zeros of the function $\zeta(s)$ in the rectangle $R(0, 1; T)$ is $N_0(\zeta(s), R(0, 1; T)) \asymp T \log T$.*

As a consequence of von Mangoldt's theorem, we have a new proof of the following theorem of Allouche [2] (see also [3]).

Theorem 5.27 (Allouche [2]). *The sequence $(\mu(n))_{n \geq 1}$ is not k -automatic for any $k \geq 2$. Hence the series $\sum_{n \geq 1} \mu(n)X^n$ is transcendental over both $\mathbb{F}_p(X)$, for all primes p , and $\mathbb{Z}(X)$.*

Proof. From the relationship

$$\sum_{n \geq 1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad (\Re s > 1),$$

for the result, we need only show that

$$\lim_{T \rightarrow \infty} \frac{1}{T} N_\infty \left(\frac{1}{\zeta(s)}, R(0, 1 : T) \right) = \infty.$$

This is given by Theorem 5.26. Application of Proposition 5.25 proves the theorem. \square

It is note-worthy that our proof for $\mu(n)$, and the proof for $|\mu(n)|$ below, does not use Cobham's theorem [26] on rational densities: *if a sequence is k -automatic for some $k \geq 2$, then the density, provided it exists, of the occurrence of any value of that sequence is rational.*

In a similar fashion to the above results, using the extension to Dirichlet L -functions of von Mangoldt's theorem, we may generalize this result further.

Lemma 5.28. *We have $N_0(L(s, \chi), R(0, 1; T)) \asymp T \log T$.*

Corollary 5.29. *Let χ be a Dirichlet character. Then $(\mu(n)\chi(n))_{n \geq 1}$ is not k -automatic for any $k \geq 2$.*

Proof. This follows directly from the fact that the sequence $(\mu(n)\chi(n))_{n \geq 1}$ is the sequence of coefficients of the series $\frac{1}{L(s, \chi)}$. Application of Lemma 5.28 and Proposition 5.25 gives the desired result. \square

The proof of Theorem 5.12 rests on substantially more than the previous results of this investigation; it requires both the Prime Number Theorem in the form below as well as a very deep result of Selberg.

Theorem 5.30 (Hadamard [53], de la Vallée Poussin [32]). *The Riemann zeta function has no zeros on the line $\Re s = 1$.*

Theorem 5.31 (Selberg [89]). *A positive proportion of the zeros of the Riemann zeta function lie on the line $\Re s = \frac{1}{2}$.*

Proof of Theorem 5.12. We use the identity

$$\mathcal{L}(s) := \sum_{n \geq 1} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \quad (\Re s > 1).$$

Using this identity, the poles of $\mathcal{L}(s)$ are precisely the zeros of $\zeta(s)$ that are not cancelled by the zeros of $\zeta(2s)$ as well as the pole of $\zeta(2s)$ at $s = \frac{1}{2}$. Selberg's theorem gives a positive proportion of zeros of $\zeta(s)$ on the critical line and the Prime Number Theorem tells us that there are no zeros on the line $\Re s = 1$. By Theorem 5.26, we have

$$N_\infty \left(\frac{\zeta(2s)}{\zeta(s)}, R \left(\frac{1}{2}, \frac{1}{2}; T \right) \right) \asymp T \log T.$$

Application of Proposition 5.25 gives the result. \square

Invoking a stronger form of Selberg's theorem, we may include many more number-theoretic functions in our investigation.

Theorem 5.32 (Conrey [27]). *More than two-fifths of the zeros of the Riemann zeta function lie on the critical line.*

Conrey's theorem gives the following corollary.

Corollary 5.33. *Less than three-tenths of the zeros of the Riemann zeta function lie on any line $\Re s = \alpha$ when $\alpha \neq \frac{1}{2}$.*

Proof. Recall that if $\zeta(s) = 0$, then by the functional equation $\zeta(1-s) = 0$. The corollary then follows from the elementary observation that $2 \cdot \frac{3}{10} + \frac{2}{5} = 1$. \square

Recall that for $m \geq 2$

$$q_m(n) = \begin{cases} 0 & \text{if } p^m \mid n \text{ for any prime } p \\ 1 & \text{otherwise.} \end{cases}$$

Hence $q_2(n) = |\mu(n)|$.

Theorem 5.34. *For $k \geq 2$, the functions $q_m(n)$ ($m \geq 2$) are not k -automatic, and hence for each $m \geq 2$ the series $\sum_{n \geq 1} q_m(n)X^n$ is transcendental over both $\mathbb{F}_p(X)$, for all primes p , and $\mathbb{Z}(X)$.*

Proof. Note the identities for $\Re s > 1$:

$$\sum_{n \geq 1} \frac{q_m(n)}{n^s} = \frac{\zeta(s)}{\zeta(ms)} \quad (m \geq 2).$$

Our result relies on $\zeta(s)/\zeta(ms)$ for each $m \geq 2$ having more than $O(T)$ poles in some rectangle. Corollary 5.33 gives

$$N_\infty \left(\frac{\zeta(s)}{\zeta(ms)}, R \left(\frac{1}{2m}, \frac{1}{2m}; T \right) \right) \asymp T \log T.$$

Application of Proposition 5.25 finishes the proof. \square

Note that the result for $|\mu(n)|$ is already given by that of $\mu(n)$ by simply defining Φ to be the absolute value function and applying Remark 5.13.

5.4 Dirichlet series and (non)regularity

We take the following definition from [6].

Definition 5.35. We say that a sequence $\mathbf{S} := (s(n))_{n \geq 0}$ taking values in a \mathbb{Z} -module R is a k -regular sequence, or just k -regular, provided there exist a finite number of sequences over R , $\{(s_1(n))_{n \geq 0}, \dots, (s_s(n))_{n \geq 0}\}$, with the property that every sequence in the k -kernel of \mathbf{S} is a \mathbb{Z} -linear combination of the s_i .

Compared to the finiteness of the k -kernel in the case of a k -automatic sequence, the above definition tells us that the sequence \mathbf{S} is k -regular provided the k -kernel of \mathbf{S} is finitely generated.

Using this definition, let $k \geq 2$ and $(v(n))_{n \geq 1}$ be a k -regular sequence with values in \mathbb{C} . Then similar to the automatic case, there exist an integer $t \geq 1$ and a sequence $(V_n)_{n \geq 1}$ with values in \mathbb{C}^t (which we denote as a column vector) as well as k matrices of size $t \times t$ denoted by B_1, B_2, \dots, B_k with integer entries (no longer just 1s and 0s as in the automatic case), such that the first component of the vector $(V_n)_{n \geq 1}$ is the sequence $(v(n))_{n \geq 1}$ and for each $i = 1, 2, \dots, k$, and for all $n \geq 1$, we have

$$V_{kn+i} = B_i V_n.$$

These properties give the analogue of Theorem 5.19 to k -regular sequences. This result is alluded to in [5, Remark 4].

Theorem 5.36. *Let $k \geq 2$ be an integer and let $(v(n))_{n \geq 0}$ be a k -regular sequence with values in \mathbb{C} . Then the Dirichlet series $\sum_{n \geq 1} v(n)n^{-s}$ is the first component of a Dirichlet vector (i.e., a vector of Dirichlet series) $G(s)$, where G has an analytic continuation to a meromorphic function on the whole complex plane, whose poles, if any, are located on a finite number of left semi-lattices.*

The proof of this theorem is exactly that of Theorem 5.19 with V_i and B_i substituted for U_i and A_i , respectively, for each i .

We now have the same useful corollaries that we had for k -automatic sequences.

Corollary 5.37. *Let $k \geq 2$. The following properties hold:*

- (i) *If the Dirichlet series $\sum_{n \geq 1} f(n)n^{-s}$ is not meromorphically continuable to the whole complex plane then $(f(n))_{n \geq 1}$ is not k -regular.*
- (ii) *If $G(s) = \sum_{n \geq 1} g(n)n^{-s}$ ($\Re s > \alpha$ for some $\alpha \in \mathbb{R}$) is meromorphically continuable to a region containing a rectangle $R(a, b, T)$ for which*

$$\lim_{T \rightarrow \infty} \frac{1}{T} N_{\infty}(G(s), R(a, b, T)) = \infty,$$

then $(g(n))_{n \geq 1}$ is not k -regular.

Theorem 5.38. *The function $\varphi(n)$ is not k -regular for any $k \geq 2$.*

Proof. From the relationship

$$\sum_{n \geq 1} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \quad (\Re s > 2),$$

and the lack of zeros of $\zeta(s-1)$ in the region $0 \leq \Re s \leq 1$ as given by the Prime Number Theorem, we need only show that

$$\lim_{T \rightarrow \infty} \frac{1}{T} N_{\infty} \left(\frac{1}{\zeta(s)}, R(0, 1 : T) \right) = \infty.$$

This is given by Theorem 5.26. Application of the Corollary 5.37 proves the theorem. \square

Theorem 5.39. *For $k \geq 2$, the functions $\rho(n)$, $\tau(n^2)$ and $\tau^2(n)$ are not k -regular.*

Proof. Note the identities for $\Re s > 1$:

$$\sum_{n \geq 1} \frac{\rho(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}, \quad \sum_{n \geq 1} \frac{\tau(n^2)}{n^s} = \frac{\zeta^3(s)}{\zeta(2s)}, \quad \sum_{n \geq 1} \frac{\tau^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}.$$

Our result relies on $\zeta(s)/\zeta(2s)$ having more than $O(T)$ poles in some rectangle. This follows directly from the proof of Theorem 5.34. \square

Theorem 5.40. *The functions $\omega(n)$ and $\Omega(n)$ are not k -regular for any $k \geq 2$.*

Proof. This follows from our proof of Theorem 5.16 and the identities

$$\sum_{n \geq 1} \frac{\omega(n)}{n^s} = \zeta(s) \sum_{k \geq 1} \frac{\mu(k)}{k} \log \zeta(ks), \quad \text{and} \quad \sum_{n \geq 1} \frac{\Omega(n)}{n^s} = \zeta(s) \sum_{k \geq 1} \frac{\varphi(k)}{k} \log \zeta(ks)$$

and the added stipulation that there are no zeros of $\zeta(s)$ on the line $\Re s = 0$. This is provided for by the Prime Number Theorem and the symmetry of zeros of the Riemann zeta function about the critical line as given by the functional equation for $\zeta(s)$. \square

Some of these results can be found from another direction using our knowledge of their non-automaticity and the following theorem (see Chapter 16 of [6] for details).

Theorem 5.41 (Allouche and Shallit [6]). *If the integer sequence $(f(n))_{n \geq 0}$ is k -regular, then for all integers $m \geq 1$, the sequence $(f(n) \bmod m)_{n \geq 0}$ is k -automatic.*

Thus if there exists an $m \geq 1$ for which $(f(n) \bmod m)_{n \geq 0}$ is not k -automatic, then $(f(n))_{n \geq 0}$ is not k -regular. Hence the results of the previous sections give non-regularity results for each of ω , Ω , τ , and ρ . It is also worth noting that a sequence is k -regular and takes on only finitely many values if and only if it is k -automatic [6]. This provides a nice relationship for non-regularity results for characteristic functions like q_m ($m \geq 2$), χ_P , and χ_{Π} .

Remark 5.42. All zeta quotient identities as well as the properties of the Riemann zeta function that were used in this chapter can be found in Titchmarsh's monograph [92].

Remark 5.43. While many of the results of this chapter have been shown before, the use of Dirichlet series to get at the non-automaticity of multiplicative functions has, with the exception of one paper [5], been ignored. This is somewhat surprising, as it is evident from the proofs given here that the systematic use of the properties of Dirichlet series,

and in particular the properties of the Riemann zeta function, along with the vast body of knowledge concerning the distribution of primes, yields a number of results concerning automaticity and regularity of number-theoretic functions. It is hoped that this method will be taken more advantage of in the future.

Chapter 6

Possible future directions

Strewn throughout this thesis there are many natural questions which arise and warrant consideration. Many of these questions appear very difficult while others will just take more time. In this chapter, we have gathered many of these questions with the hope of presenting a clearer picture of where future work surrounding these topics may lead.

6.1 Sums of multiplicative functions

Throughout Chapter 2 we were interested in estimates concerning the partial sums of multiplicative functions. An important question is: *what can be said about the growth of $\left| \sum_{n \leq x} f(n) \right|$ for any function $f \in \mathcal{F}(\{-1, 1\})$?* This question goes back to Erdős [45]. He states

Finally, I would like to mention an old conjecture of mine: let $f(n) = \pm 1$ be an arbitrary number-theoretic function. Is it true that to every c there is a d and an m so that

$$\left| \sum_{k=1}^m f(kd) \right| > c?$$

I have made no progress with this conjecture.

Concerning this conjecture, he adds in [46]

The best we could hope for is that

$$\max_{md \leq n} \left| \sum_{k=1}^m f(kd) \right| > c \log n.$$

We remark that these questions can also be asked for functions $f(n)$ which take k th roots of unity as values rather than just ± 1 . However, very little is yet known for this case.

Our question only pertains to completely multiplicative functions, and in this setting it may be possible to answer Erdős' conjecture. It seems that automatic functions in $\mathcal{F}(\{-1, 1\})$ will give the smallest growth of partial sums. Recall that for p an odd prime and λ_p a character-like function, we have

$$\max_{N \leq x} \left| \sum_{n \leq N} \lambda_p(n) \right| \asymp \log x.$$

Our investigations lead us to conjecture the following.

Conjecture 6.1. *Let $f \in \mathcal{F}(\{-1, 1\})$ be k -automatic for some $k \geq 2$. If $\sum_{n \leq x} f(n) = o(x)$, then $\max_{N \leq x} \left| \sum_{n \leq N} f(n) \right| \asymp \log x$.*

6.2 Algebraic character of generating functions

Many of the results of this thesis concern the algebraic character of the generating series of a multiplicative function. The usual result is to choose a multiplicative function $f : \mathbb{N} \rightarrow R$ and show that the series $\sum_{n \geq 1} f(n)x^n$ is transcendental over $R(x)$. For example, in the case of Theorem 5.12 the function $f(n) = \lambda(n)$ and $R = \mathbb{F}_p$ for any $p > 2$. Results like this lead us to the following question.

Question 6.2. Let R be a ring and $f : \mathbb{N} \rightarrow R$ be a multiplicative function. Suppose that $\sum_{n \geq 1} f(n)x^n$ is algebraic over $R(x)$. What can be said about $f(n)$?

In the case that R has characteristic zero, the result is known. Bell and Coons [10], have shown that if R is of characteristic zero and $\sum_{n \geq 1} f(n)x^n$ is algebraic over $R(x)$, then either there is a natural number k and a periodic multiplicative function $\chi(n)$ such that $f(n) = n^k \chi(n)$ for all n , or $f(n)$ is eventually zero.

The question is much more interesting when we consider R to be of positive characteristic. No progress has been made in this case, and to the best of our knowledge there are no known conjectures regarding the case of positive characteristic.

6.3 Transcendence and functional equations

Theorem 3.13 gives that a non-zero power series $F(z) \in \mathbb{C}[[z]]$ satisfying

$$F(z^d) = F(z) + \frac{A(z)}{B(z)}, \quad (6.1)$$

where $A(z), B(z) \in \mathbb{C}[z]$ with $A(z) \neq 0$ and $\deg A(z), \deg B(z) < d$ is transcendental over $\mathbb{C}(z)$.

One should have a similar result for a more general functional equation like

$$F(z^d) = \frac{A_n(z)}{B_n(z)}F(z)^n + \frac{A_{n-1}(z)}{B_{n-1}(z)}F(z)^{n-1} + \cdots + F(z) + \frac{A_1(z)}{B_1(z)}F(z) + \frac{A_0(z)}{B_0(z)}.$$

The method of proof of Theorem 3.13 should apply to this case as well. In place of binomial coefficients one would have multinomial coefficients which will make the combinatorics involved much more complicated. We have attempted the case $n = 2$, but abandoned it since the details were so messy.

6.4 Transcendental values of series

As the results in Chapter 4 demonstrate, transcendence results on power series are readily available, though there are many open questions concerning their special values. The question we would most like to answer is that of the transcendence of

$$\sum_{n \geq 1} \frac{\lambda(n)}{2^n},$$

though there are many other open questions concerning the algebraic character of values of generating functions of multiplicative functions.

Erdős [43] was interested in the transcendence, and irrationality, of both $\sum_{n \geq 1} \tau(n)2^{-n}$ and $\sum_{n \geq 1} \varphi(n)2^{-n}$. The irrationality of the first sum was shown by Borwein [11]; the transcendence is not yet known. Questions regarding the algebraic character of $\sum_{n \geq 1} \varphi(n)2^{-n}$ remain open. Nesterenko [79] has shown that the number

$$\sum_{n \geq 1} \frac{\lambda(n)}{2^n - 1}$$

is transcendental, but his method does not seem to generalize to help with the case of $\sum_{n \geq 1} \lambda(n)2^{-n}$.

6.5 Correlation and diversity

There is much to do concerning the correlation and diversity of arithmetic functions, and it seems that the available methods and results leave many ideas ripe for development.

Concerning transcendence of power series of generating functions, one need not dig so deeply to give transcendence results over $\mathbb{Z}(X)$ or $\mathbb{Q}(X)$ using theorems like the following.

Theorem 6.3 (Fatou [48]). *If $F(X) = \sum_{n \geq 1} f(n)X^n \in \mathbb{Z}[[X]]$ converges inside the unit disk, then either $F(X) \in \mathbb{Q}(X)$ or $F(X)$ is transcendental over $\mathbb{Q}(X)$.*

Carlson [21], proving a conjecture of Pólya, added to Fatou's theorem.

Theorem 6.4 (Carlson [21]). *A series $F(X) = \sum_{n \geq 1} f(n)X^n \in \mathbb{Z}[[X]]$ that converges inside the unit disk is either rational or it admits the unit circle as a natural boundary.*

Recall that if $f(n) = O(n^d)$ for some d , the series $F(X) = \sum_{n \geq 1} f(n)X^n \in \mathbb{Z}[[X]]$ has radius of convergence 1, so that by Carlson's Theorem, such a series is transcendental over $\mathbb{Q}(X)$. This gives very quick transcendence results for series $F(X)$ with $f(n) = \varphi(n), \tau(n^2), \tau^2(n), \omega(n)$, and $\Omega(n)$. Noting that by the Prime Number Theorem, $p(n) \sim n \log n = O(n^2)$, we have the following result for the n th prime number.

Proposition 6.5. *The series $\sum_{n \geq 1} p(n)X^n \in \mathbb{Z}[[X]]$ is transcendental over $\mathbb{Q}(X)$, and hence also over $\mathbb{Z}(X)$.*

The ideas of k -regularity may be exploitable to give transcendence results using the following theorem of Allouche and Shallit from [6] and a combination of the above theorems in this section, though it seems at this point that a case by case analysis would be necessary, which we believe would not make for easy reading.

Theorem 6.6 (Allouche and Shallit [6]). *Let K be an algebraically closed field (e.g., \mathbb{C}). Let $(s(n))_{n \geq 0}$ be a sequence with values in K . Let $S(X) = \sum_{n \geq 0} s(n)X^n$ be a formal power series in $K[[X]]$. Assume that S represents a rational function of X . Then $(s(n))_{n \geq 0}$ is k -regular if and only if the poles of S are roots of unity.*

One may be able to form this into more rigid and inclusive theorems and as such, this seems a worthy endeavor.

Concerning more specific functions, the non-automaticity of $\lambda(n)$ (and similarly $\mu(n)$) is somewhat weak compared to the expected properties of the correlation. One expects that

for any $A, B, a, b \in \mathbb{N}$ with $aB \neq Ab$

$$\left| \sum_{n \leq x} \lambda(An + B)\lambda(an + b) \right| = o(x),$$

so that not only should the k -kernel be infinite, as shown in this chapter, but no two sequences of λ -values on distinct arithmetic progressions, which are not multiples of each other, should be equal. In this sense, the Liouville function should be a sort of “worst case scenario” for non-automaticity concerning multiplicative functions. To make this more formal, consider the idea of *diversity* as introduced by Shallit [90, Section 5].

A sequence is said to be k -diverse if every sequence in the k -kernel is distinct. Since for a completely multiplicative function f , we have $f(kn) = f(k)f(n)$ for all k . If there is a $k > 1$ for which $f(k) = 1$, then f is identical on the two arithmetic progressions kn and n , and hence the sequence of values of such an f cannot be k -diverse; there is such a k for λ . This case can be excluded. The following definition is taken from [90].

Definition 6.7. A sequence $(s(i))_{i \geq 0}$ is *weakly k -diverse* if the $\varphi(k)$ subsequences $\{(s(ki + a))_{i \geq 0} : \gcd(a, k) = 1, 1 \leq a < k\}$ are all distinct. A sequence is *weakly diverse* if it is weakly k -diverse for all $k \geq 2$.

Using this language, we finish with the following conjecture.

Conjecture 6.8. *The sequence $(\lambda(n))_{n \geq 1}$ is weakly diverse.*

We can presently think of two possible attacks on this conjecture. Firstly, we can try to find solutions to the Diophantine equations

$$(p^{l_1}x + r_1)(p^{l_2}x + r_2) = cy^2 \tag{6.2}$$

where p is a given prime, $l_1, l_2 \in \mathbb{N} \cup \{0\}$, and we may take a c for which $\lambda(c) = 1$ and a c for which $\lambda(c) = -1$. As a first step, if we take $r_1 = r_2 = 1$, (6.2) reduces to finding solutions to

$$x_1^2 - c_1y^2 = d,$$

where

$$x_1 := \left(p^{l_1+l_2}x + \frac{p^{l_1} + p^{l_2}}{2} \right), \quad c_1 := p^{l_1+l_2}c, \quad d := \frac{(p^{l_1} + p^{l_2})^2}{4} - 1.$$

A second possible attack is to try to apply Elliott’s Theorem to the correlation of $\lambda(n)$.

Theorem 6.9 (Elliott [42]). *Let $a, A > 0$ and $b, B \in \mathbb{Z}$ with $aB \neq Ab$. Let $x > 2$, $(\log x)^{-\frac{1}{100}} \leq \delta \leq \frac{1}{3}$. Let g_1 and g_2 be complex valued multiplicative functions, of absolute value at most one, which satisfy*

$$y^{-1} \left| \sum_{n \leq y} g_1(an + b)g_2(An + B) \right| \geq 1 - \delta^2$$

uniformly for $x^\delta \leq y \leq x$.

If there is a character χ modulo $a(a, b)^{-1}$, and a real τ , $|\tau| \leq x^\delta (\log x)^{-2}$ for which

$$\sum_{p \leq x} \frac{1}{p} |1 - g_1(p)\chi(p)p^{it}|^2 \leq \frac{1}{3} \log \frac{1}{\delta},$$

then there exists a real μ , $|\mu| \leq 1$, such that

$$\sum_{p \leq x} \frac{1}{p} \left(1 - \Re g_1(p)\chi(p)p^{i(\tau+\mu)}\right) \leq c_0$$

where the constant c_0 depends at most on the four integers a, A, b, B .

Just after the statement of this theorem in [42], Elliott writes that “applied to individual multiplicative functions, such as the Möbius function, Theorem 6.9 appears weak in comparison with expected results.” By this statement, one assumes that Elliott has applied this theorem to the Möbius function, though there is no record of it in the literature.

Appendix A

Proof of Mahler's Theorem

This appendix contains a proof of Mahler's Theorem [73] (Theorem 3.4 of this thesis), as taken verbatim from Nishioka's book [83]. Here \mathbf{I} is the set of algebraic integers over \mathbb{Q} , K is an algebraic number field, $\mathbf{I}_K = K \cap \mathbf{I}$, and $f(z) \in K[[z]]$ with radius of convergence $R > 0$ satisfying the functional equation for an integer $d > 1$,

$$f(z^d) = \frac{\sum_{i=0}^m a_i(z) f(z)^i}{\sum_{i=0}^m b_i(z) f(z)^i}, \quad m < d, \quad a_i(z), b_i(z) \in \mathbf{I}_K[z],$$

and $\Delta(z) := \text{Res}(A, B)$ is the resultant of $A(u) = \sum_{i=0}^m a_i(z) u^i$ and $B(u) = \sum_{i=0}^m b_i(z) u^i$ as polynomials in u . Also,

$$\overline{|\alpha|} := \max\{|\alpha^\sigma| : \sigma \in \text{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})\} \quad \text{and} \quad \text{den}(\alpha) := \min\{d \in \mathbb{Z} : d > 0, d\alpha \in \mathbf{I}\}.$$

Mahler's Theorem. *Assume that $f(z)$ is not algebraic over $K(z)$. If α is an algebraic number with $0 < |\alpha| < \min\{1, R\}$ and $\Delta(\alpha^{d^k}) \neq 0$ ($k \geq 0$), then $f(\alpha)$ is transcendental.*

Proof. Suppose that $f(\alpha)$ is algebraic. We may assume $\alpha, f(\alpha) \in K$. Let n be a positive integer. Then there are $n + 1$ polynomials $P_0, P_1, \dots, P_n \in \mathbf{I}_K[z]$ with degrees at most n such that the auxiliary function

$$E_n(z) = \sum_{j=0}^n P_j(z) f(z)^j = \sum_{h \geq 0} b_h z^h$$

is not identically zero and all the coefficients b_h , with $h < n^2$, vanish. Since $f(z)$ is not algebraic over $K(z)$, $E_n(z)$ is not identically zero. Let H be the least integer such that $b_H \neq 0$. Then $H > n^2$. Since

$$\lim_{z \rightarrow 0} E_n(z) z^{-H} = b_H,$$

we have for any $k \geq c_1(n)$,

$$0 \neq |E_n(\alpha^{d^k})| \leq c_2(n)|\alpha|^{d^k H} \leq c_2|\alpha|^{d^k n^2}. \quad (\text{A.1})$$

There are polynomials $S(z, u), T(z, u) \in \mathbf{I}_K[z, u]$ such that

$$\Delta(z) = S(z, u) \sum_{i=0}^m a_i(z) u^i + T(z, u) \sum_{i=0}^m b_i(z) u^i.$$

Hence

$$\Delta(\alpha) = S(\alpha, f(\alpha)) \sum_{i=0}^m a_i(\alpha) f(\alpha)^i + T(\alpha, f(\alpha)) \sum_{i=0}^m b_i(\alpha) f(\alpha)^i.$$

Suppose that $\sum_{i=0}^m b_i(\alpha) f(\alpha)^i = 0$. Since

$$\left(\sum_{i=0}^m b_i(\alpha) f(\alpha)^i \right) f(\alpha^d) = \sum_{i=0}^m a_i(\alpha) f(\alpha)^i,$$

we get $\sum_{i=0}^m a_i(\alpha) f(\alpha)^i = 0$ and so $\Delta(\alpha) = 0$. This contradicts the assumption. Therefore $\sum_{i=0}^m b_i(\alpha) f(\alpha)^i \neq 0$ and $f(\alpha^d) \in K$. preceding in this way, we see that $f(\alpha^{d^k}) \in K$ and therefore $E_n(\alpha^{d^k}) \in K$ ($k \geq 0$). Define Y_k ($k \geq 0$) inductively as follows,

$$Y_1 = \sum_{i=0}^m b_i(\alpha) f(\alpha)^i,$$

$$Y_{k+1} = Y_k^m \sum_{i=0}^m b_i(\alpha^{d^k}) f(\alpha^{d^k})^i, \quad k \geq 1.$$

Then $Y_k \in K$ and $Y_k \neq 0$ ($k \geq 0$). We estimate $|\overline{Y_k^n E_n(\alpha^{d^k})}|$ and $\text{den}(Y_k^n E_n(\alpha^{d^k}))$. Let $\deg_z(b) i(z) \leq l, |\overline{\alpha}|, |\overline{f(\alpha)}| \leq c_3$ ($c_3 > 1$) and D a positive integer such that $D\alpha, Df(\alpha) \in \mathbf{I}$. Then we have

$$|\overline{Y_1}| = \overline{\left| \sum_{i=0}^m b_i(\alpha) f(\alpha)^i \right|} = \sum_{i=0}^m |\overline{b_i(\alpha)}| |\overline{f(\alpha)}|^i \leq c_4 c_3^l c_3^m,$$

$$|\overline{Y_1 f(\alpha^d)}| = \overline{\left| \sum_{i=0}^m a_i(\alpha) f(\alpha)^i \right|} = \sum_{i=0}^m |\overline{a_i(\alpha)}| |\overline{f(\alpha)}|^i \leq c_4 c_3^l c_3^m$$

and

$$D^{l+m} Y_1, D^{l+m} Y_1 f(\alpha^d) \in \mathbf{I}.$$

Since $Y_2 = Y_1^m \sum_{i=0}^m b_i(\alpha^d) f(\alpha^d)^i$ and $Y_2 f(\alpha^{d^2}) = Y_1^m \sum_{i=0}^m a_i(\alpha^d) f(\alpha^d)^i$, we have

$$|\overline{Y_2}|, |\overline{Y_2 f(\alpha^{d^2})}| \leq (c_4 c_3^{dl}) (c_4 c_3^{l+m})^m$$

and

$$D^{dl}(D^{l+m})^m Y_2, D^{dl}(D^{l+m})^m Y_2 f(\alpha^{d^2}) \in \mathbf{I}.$$

preceding in this way, we obtain

$$\overline{|Y_k|}, \overline{|Y_k f(\alpha^{d^k})|} \leq c_4^{1+m+\dots+m^{k-1}} (c_3^l)^{d^{k-1}+d^{k-2}m+\dots+m^{k-1}} c_3^{m^k}$$

and

$$\begin{aligned} (D^l)^{d^{k-1}+d^{k-2}m+\dots+m^{k-1}} D^{m^k} Y_k &\in \mathbf{I}, \\ (D^l)^{d^{k-1}+d^{k-2}m+\dots+m^{k-1}} D^{m^k} Y_k f(\alpha^{d^k}) &\in \mathbf{I}. \end{aligned}$$

By the assumption $m < d$, we have

$$d^{k-1} + d^{k-2}m + \dots + m^{k-1} = d^{k-1} \left(1 + \frac{m}{d} + \dots + \left(\frac{m}{d}\right)^{k-1} \right) \leq c_5 d^{k-1},$$

where we take a positive integer as c_5 . Hence

$$\overline{|Y_k|}, \overline{|Y_k f(\alpha^{d^k})|} \leq c_4^{c_5 d^{k-1}} (c_3^l)^{c_5 d^{k-1}} c_3^{d^k} \leq c_6^{d^k}$$

and

$$D_0^{d^k} Y_k, D_0^{d^k} Y_k f(\alpha^{d^k}) \in \mathbf{I}, \quad D_0 = D^{lc_5+1}.$$

Since

$$Y_k^n E_n(\alpha^{d^k}) = \sum_{j=0}^n P_j(\alpha^{d^k}) Y_k^{n-j} \left(Y_k f(\alpha^{d^k}) \right)^j,$$

we obtain

$$\overline{|Y_k^n E_n(\alpha^{d^k})|} \leq c_7(n) c_3^{d^k n} c_6^{d^k n}, \quad D_0^{2d^k} Y_k^n E_n(\alpha^{d^k}) \in \mathbf{I}. \quad (\text{A.2})$$

By (A.1), (A.2) and the fundamental inequality,

$$\begin{aligned} d^k n \log c_6 + \log c_2(n) + d^k n^2 \log |\alpha| &\geq \log |Y_k^n E_n(\alpha^{d^k})| \\ &\geq -2[K : \mathbb{Q}] (\log c_7(n) + d^k n \log c_3 c_6 + 2d^k n \log D_0), \end{aligned}$$

for $k > c_1(n)$. Dividing both sides above by d^k and letting k tend to infinity, we have

$$n \log c_6 + n^2 \log |\alpha| \geq -2[K : \mathbb{Q}] (n \log c_3 c_6 + 2n \log D_0).$$

Dividing both sides above by n^2 and letting n tend to infinity, we have $\log |\alpha| \geq 0$, a contradiction. \square

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